CFT_4's and $CCFT_2$

(A slight free association)

Based on future mystery papers? with Tianli and Jordan

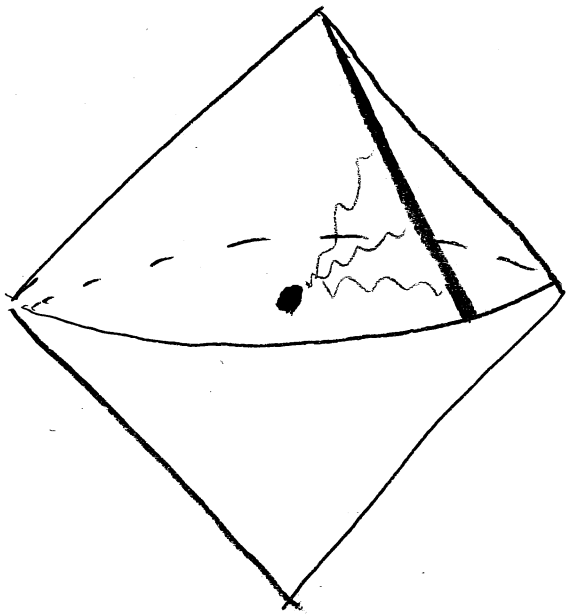
Noah Miller

simons collaboration on Celestial Holography workshop, Harvard University Oct 26, 2026

Recent exciting results on CFT's in celestial holography.

For instance, careful study of light-ray operators implies that $L_{w,+\infty}$ is symmetry of any CFT_4 (Mina + Monica)

(see also Ahmed, Andy, Hongji, Hernán, Jakob, Brett, Ian, Sabrina ...)



Light-ray operator + local operator set up has historically been studied in "Conformal Colliders" story (Hofman + Maldacena '08)

A question one may have: DO CFT's HAVE S-MATRICES?

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}} \xrightarrow{\text{Fourier}} \langle \tilde{\mathcal{O}}(k)\tilde{\mathcal{O}}(0)\rangle \sim \frac{1}{|k|^{d-2\Delta}}$$

LSZ: Amplitudes = Amputated correlators

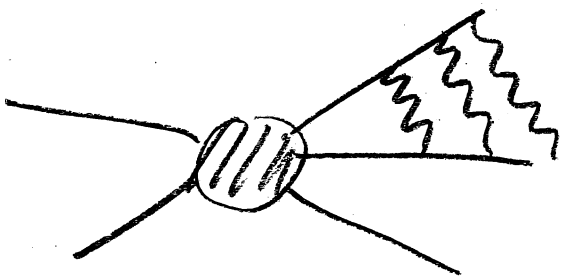
(multiply by $k^2 \to 0$, cancel pole)

In general, CFT 2-pt functions are not a discrete sum of poles, amplitudes vanish if $\Delta > \frac{d-2}{2}$.

Nonetheless, $N=4$ SYM is a CFT_4, and there are about 8.5 trillion papers written about its amplitudes. So what gives?

Well, $N=4$ SYM amplitudes do techically vanish, but for a boring reason, due to infrared divergences.

This is the case for any theory with massless particles.



$\sim e^{-g^{2\infty}}$

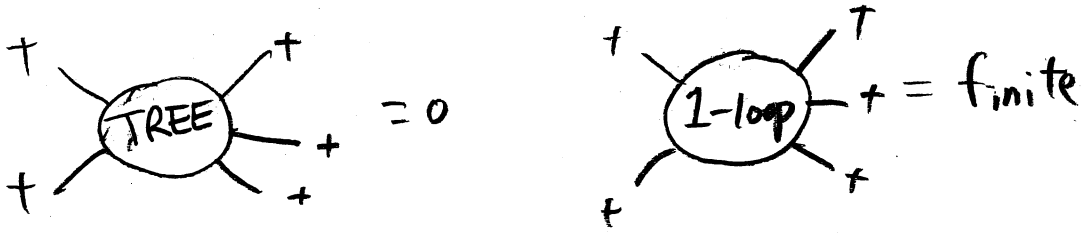
Actually, there is one CFT_4 I am aware of with finite amplitudes: pure SDYM.

Tree-level-all-plus and Tree-level-one-minus vanish

One-loop all plus finite ←

Bulk $SO(4,2)$ conformally invariant, Henn, Power, Zoia '19

One-loop-exact, no exponentiation!



Presumably exploits a loophole due to $\frac{\varepsilon}{\varepsilon}$ cancellation!

But I have nothing more to say about this...

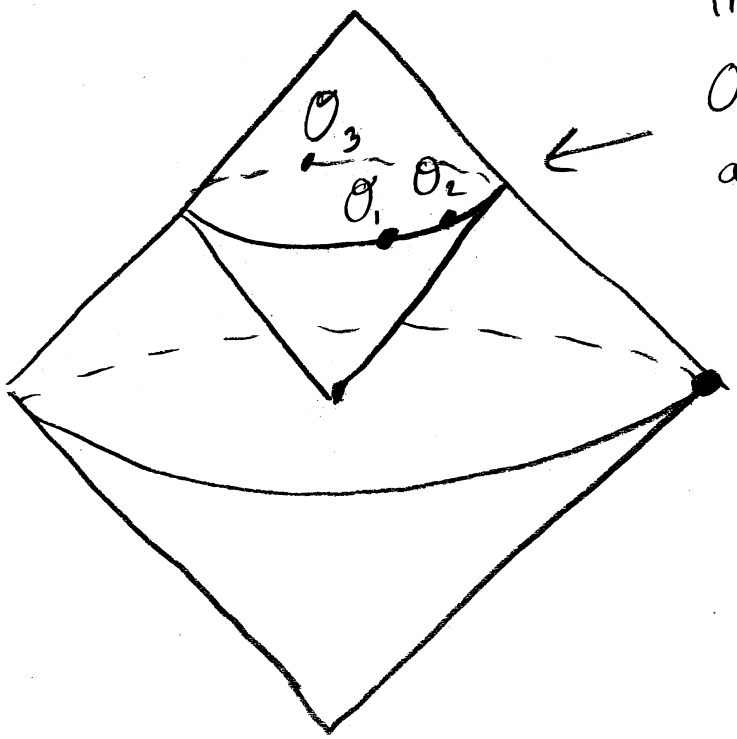
Actually, only CFT_4 perturbatively, broken by instanton corrections
Kevin, Roland '25

Question:

Is there some notion of a "soft Theorem" for position space CFT correlation functions?

(I have nothing interesting to say about light-ray operators.)

First thought...

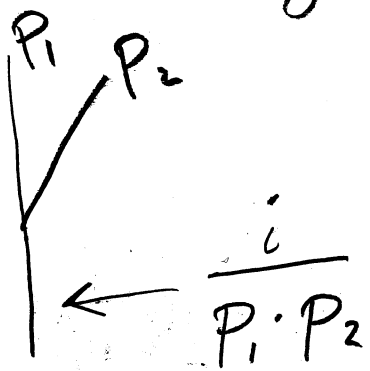


Intuitively...

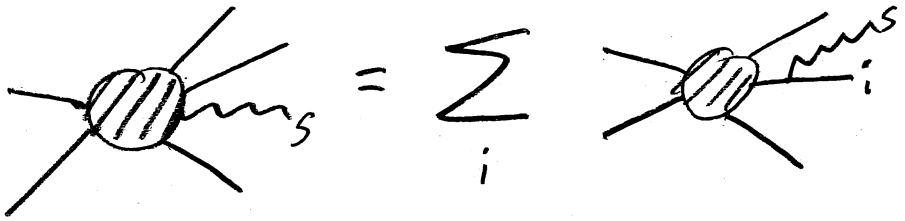
Operators on null cone at $\mathcal{I}^+$ are like "hard particles"

$\mathcal{O}_S$ ← Operator at spatial infinity should be asymptotic symmetry or "soft insertion"

Second Thought...



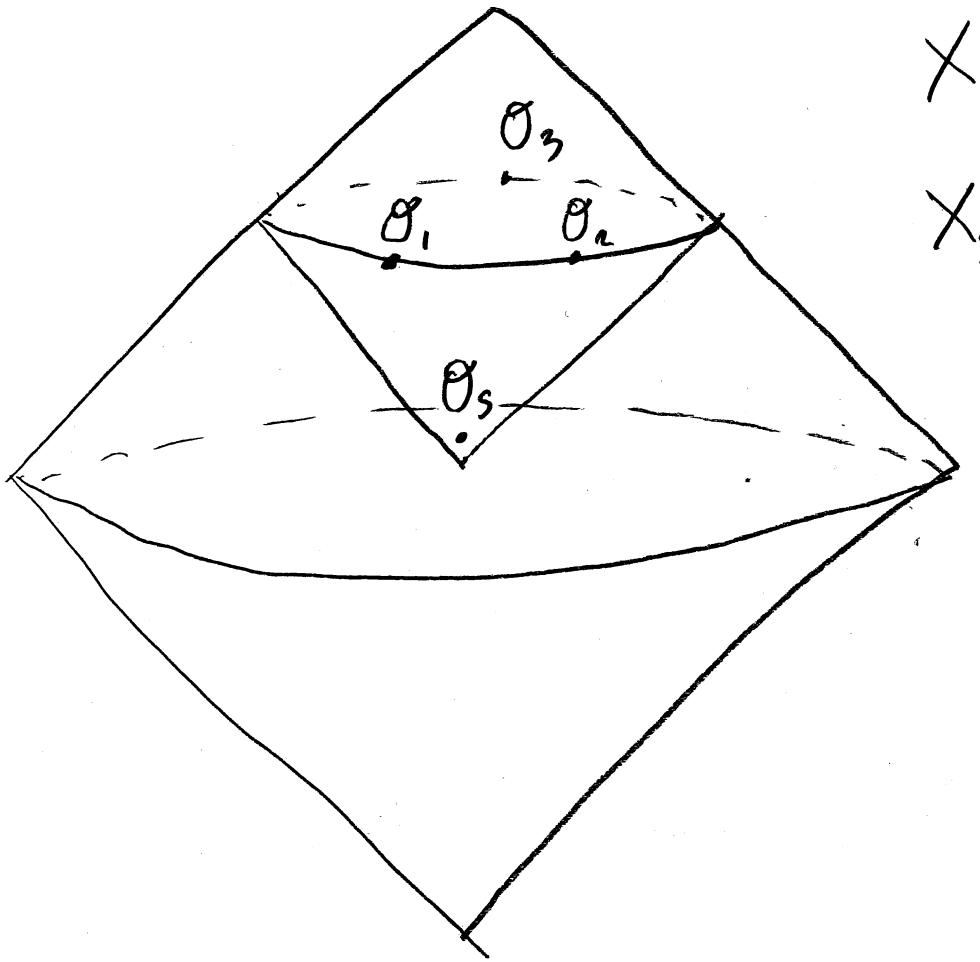
Holomorphic collinear limit:
single out $p_1 \cdot p_2 \to 0$ pole



Soft limit:
simultaneous $p_s \cdot p_i \to 0$
limit for all p_i,
get sum over external legs

Position-Space analog of "SOFT LIMIT?"

POSITION-SPACE ANALOG of "SOFT-LIMIT?"

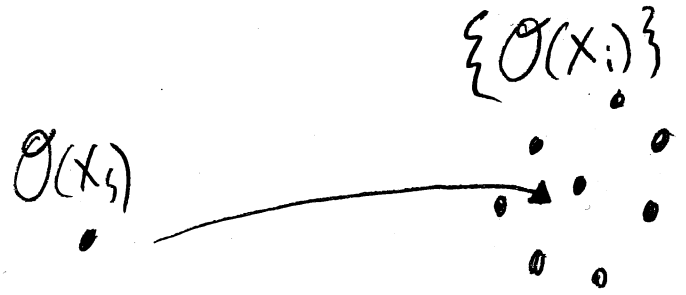


X_i on lightcone at $\mathcal{I}^+$

X_s near origin

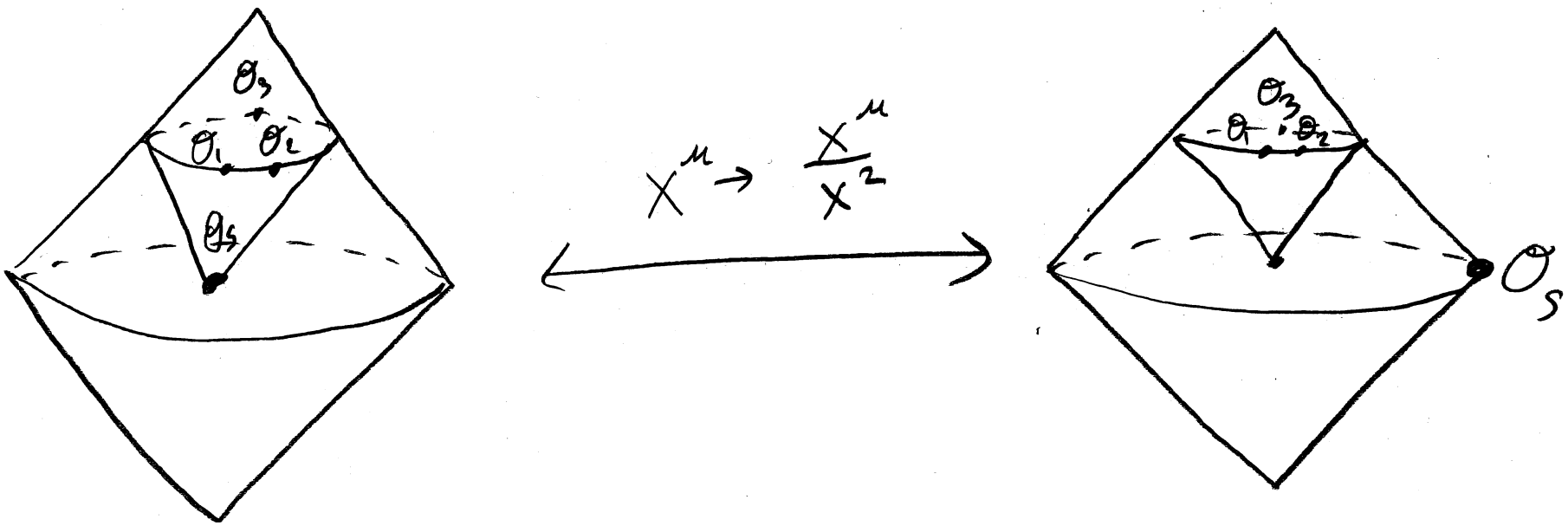
$$(X_i - X_s)^2 \to 0$$

Note that such a configuration isn't possible in Euclidean signature



But I also don't know of any notion of a soft theorem in Euclidean signature anyway

"First thought" and "second thought" related by conformal inversion:



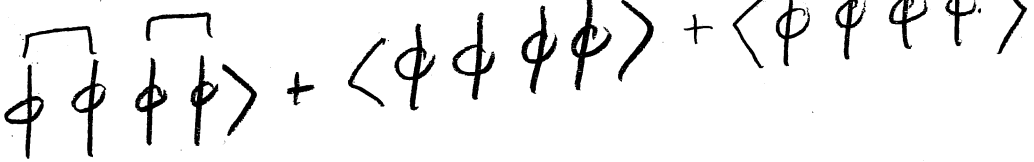
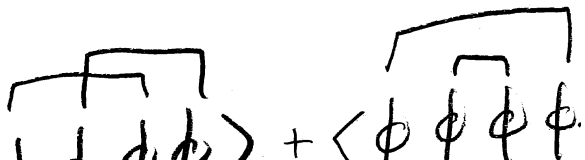
So does such a soft theorem exist?

Consider free massless scalar, CFT$_4$ with $\Delta_4 = 1$

$$S[\phi] = \frac{1}{2}\int d^4x\,(\partial\phi)^2, \qquad \langle\phi(x)\phi(y)\rangle \sim \frac{1}{(x-y)^2}$$

Only OPE: $\phi(x)\phi(y) \sim \frac{I}{(x-y)^2}$

Evaluate correlators with wick contractions:

$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle =$  $+$  $+$

(Different from OPE expansion)

In a free theory,

$$\langle \phi(X_s)\phi(X_1)\dots\phi(X_i)\dots\phi(X_n)\rangle$$

$$= \sum_{i=1}^{n} \frac{1}{-2X_i\cdot X_s} \langle \phi(X_1)\dots \underline{1} \dots \phi(X_n)\rangle$$

with $X_i \to 0$, $X_s^2 = 0$.

Here we had access to Wick contractions (Feynman Diagrams)

Seems like a "simultaneous OPE"

Having said this, I don't know how to make
an analogous statement in an interacting
theory where I don't have access to wick contractions...
Maybe a few ideas but not sure...

Nonetheless, I want to support the idea
that operators at $\mathcal{I}^+$ are "hard particles"
and operators at i^0 are "soft particles."

To do this, we will look at the "single particle wavefunctions" creates by the local operators.

Reminder:

$$(f,g)_{KG} = i\int_{\Sigma} d^3x\,(f^*\partial_0 g - g\partial_0 f^*)$$

$$= i(f^*, g)_{sym}$$

$$\hat{a}(f) \equiv (f, \hat{\phi})_{KG}$$

$$\hat{a}^\dagger(f) = -(f^*, \hat{\phi})_{KG}$$

$$[\hat{a}(f), \hat{a}^\dagger(g)] = (f,g)_{KG}$$

Example: $\hat{a}_{\vec{k}} = \hat{a}\left(e^{-ik^0x^0 + i\vec{k}\cdot\vec{x}}\right)$ for $k^0 = |\vec{k}| > 0$

For any $\Box f = 0$

$$f = f^+ + f^-$$

↑ pos. freq ↑ neg. freq

$$\hat{a}^\dagger(f^-)|0\rangle = 0$$

neg. freq. modes annihilate vacuum

Quick aside: why do negative frequency modes annihilate vacuum?

(Trick from de-Sitter paper NM '25):

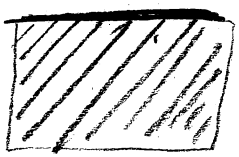
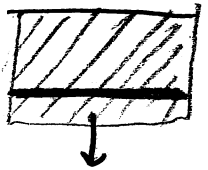
Every free theory has an infinite number of symmetries.

For every f such that $\Box f = 0$, $\phi \to \phi + \epsilon f$

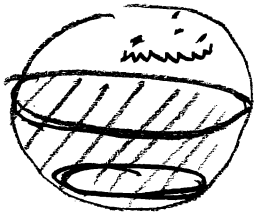
Noether charge of this symmetry is the creation operator

$$a^\dagger(f) = -i(f, \phi)_{sym}$$

FLAT SPACE:

$\hat{a}^\dagger(f^-)|0\rangle =$  Euclidean time $=$  $= 0$

de Sitter SPACE:

$\hat{a}^\dagger(f^-)|0\rangle =$  $=$  $= 0$

Bunch-Davies mode

For local free-field operator $\hat{\phi}(Y)$, we want $f_Y(X)$ so

$$\hat{\phi}(Y) = \hat{a}^\dagger(f_Y)$$

i.e. we want $K(X,Y)$

$$\hat{\phi}(Y) = -i\left(K(X,Y), \hat{\phi}(X)\right)_{sym,X}$$

such that

$$\Box_X K(X,Y) = \Box_Y K(X,Y) = 0$$

and

$$K((0,\vec{X}),(0,\vec{Y})) = 0$$

$$(\partial_{X^0} K)((0,\vec{X}),(0,\vec{Y})) = -i\delta^{(3)}(\vec{X}-\vec{Y})$$

Solution is the "Radiative propagator"

$$K(X,Y) = i\,G_{rad}(X-Y)$$
$$= \langle 0 | [\hat{\phi}(X), \hat{\phi}(Y)] | 0 \rangle$$

satisfies

$$G_{rad}(X) = \frac{1}{4\pi|\vec{x}|}\left(\delta(x^0 - |\vec{x}|) - \delta(x^0 + |\vec{x}|)\right)$$
$$= i\,G^{+}(X) - iG^{-}(X) \leftarrow \text{pos./neg. freq Wightman funcs}$$
$$= G_{ret}(X) - G_{adv}(X) \leftarrow \text{retarded/advanced propagators solve sourced } \Box G_{ret/adv} = -\delta^{(4)}(X)$$



with pos./neg. freq parts

$$G^{\pm}(X) = \frac{1}{(2\pi)^2} \frac{1}{-(x^0 \mp i\epsilon)^2 + \vec{x}^2}$$

So,

$$\hat{\phi}(Y)|0\rangle = a^{+}(f_Y^{+})|0\rangle$$

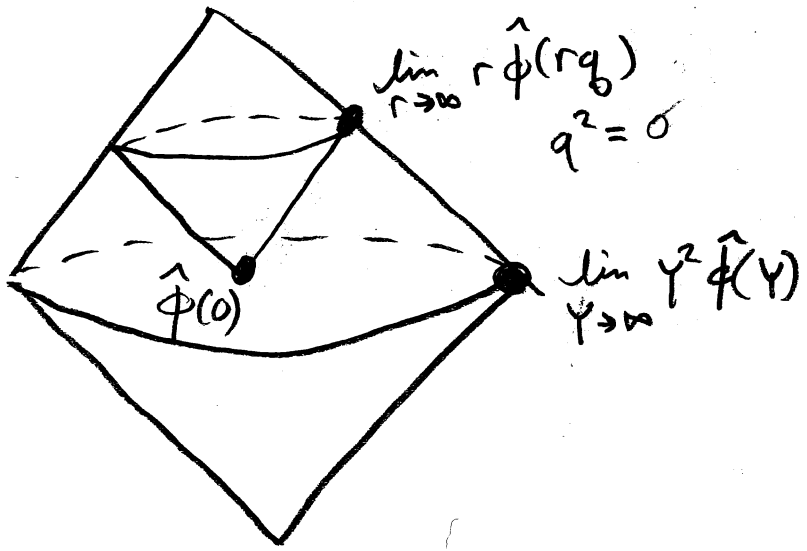
with

$$f_Y^{+}(X) = -G^{+}(X-Y)$$

$$= -\frac{1}{(2\pi)^2}\frac{1}{(X_+ - Y)^2}$$

↑ iε prescription

Now we shall compute states



Operator all $\mathcal{I}^+$:

$$\lim_{r\to\infty} r\hat{\phi}(rq)|0\rangle = \lim_{r\to\infty} r\hat{a}(f^+_{rq})|0\rangle$$

$$\lim_{r\to\infty} r f^+_{rq}(X) = \lim_{r\to\infty} \frac{r}{(X_+ - rq)^2} = \frac{1}{q\cdot X_+} = \psi^+_{\Delta=1,q}(X)$$

$\Delta=1$ "hard" particle!

Operator at origin:

$$\hat{\phi}(0)|0\rangle = \hat{a}(f_0^+)|0\rangle$$

$$f_0^+(X) = \frac{1}{X_+^2} = \widetilde{\psi^+_{\Delta=2,q}}(X)$$

Shadow $\Delta=0$ mode

Descendants:

$$(q\cdot P)^n \frac{1}{X_+^2} = \frac{(q\cdot X_+)^n}{X_+^{2n}} = \widetilde{\psi^+_{\Delta=2-n,q}(X)}$$

Shadow $\Delta=0,1,2\ldots$ mode

Operator at i^0:

$$\lim_{Y\to\infty} Y^2\phi(Y)|0\rangle = \lim_{Y\to\infty} Y^2\hat{a}^+(f^+_Y)|0\rangle$$

$$\lim_{Y\to\infty} Y^2 f^+_Y(X) = \lim_{Y\to\infty} \frac{Y^2}{(X_+ - Y)^2} = 1 = \psi^+_{\Delta=0,q}(X)$$

Descendants:

$$(q\cdot K)^n 1 = (q\cdot X_+)^n = \psi^+_{\Delta=-n,q}(X)$$

SOFT $\Delta = 0,-1,-2,\ldots$ MODES

Let me make a final comment on an older paper [NM, Jordan, Andy '23]
[NM, Erin, Sruthi, Andy '21]

$$(f,g)_{RSW} = (f,(RSW)\,g)_{KG}$$

$x^3 \to -x^3$: shadow; $\Delta \to 2-\Delta$

$$\left(L_{-1}^{n_1} \bar{L}_{-1}^{\bar{n}_1} \psi^{\pm}_{\Delta_1,N},\ L_{-1}^{n_2} \bar{L}_{-1}^{\bar{n}_2} \psi^{\pm}_{\Delta_2,N}\right) = \pm \delta_{\Delta_1\Delta_2}\,\delta_{n_1 n_2}\,\delta_{\bar{n}_1\bar{n}_2}\,M_{\Delta_1,n_1,n_2}$$

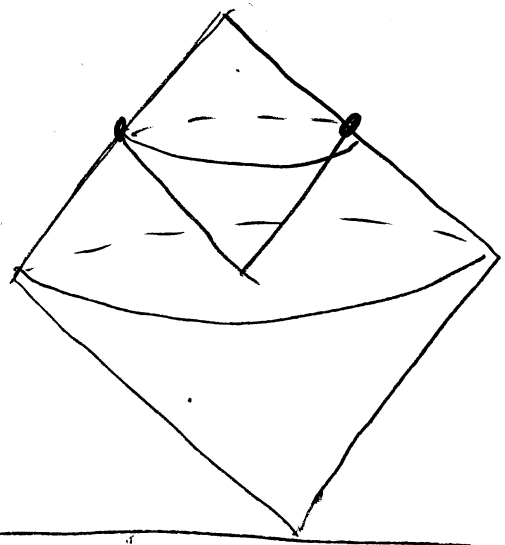
$$G^{+}(X-Y) = \sum_{\Delta\in\mathbb{Z}_{\le 0}} \sum_{n,\bar{n}=0}^{-\Delta} \frac{L_{-1}^{n}\bar{L}_{-1}^{\bar{n}}\psi^{+}_{\Delta,N}(X)\,L_1^{n}\bar{L}_1^{\bar{n}}\widetilde{\psi^{-}_{2-\Delta,S}}(Y)}{M_{-\Delta,n,n}}$$

$M_{-\Delta,n,n}$ ↖ finite

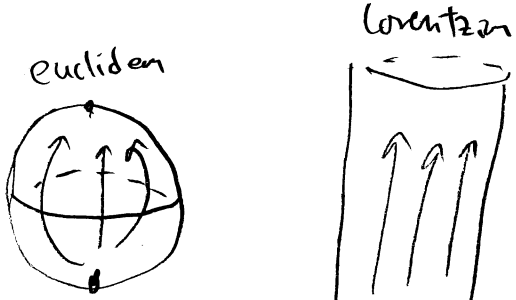
In euclidean CFT_4, one can make a similar sum with bras/kets at origin/$i\infty$



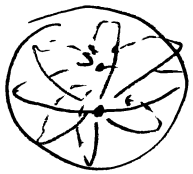
In retrospect, my inner product with Jordan/Andy is cleanly this at North Pole/South Pole (essentially)



(This is probably a "complex" notion of quantization)



standard quantization



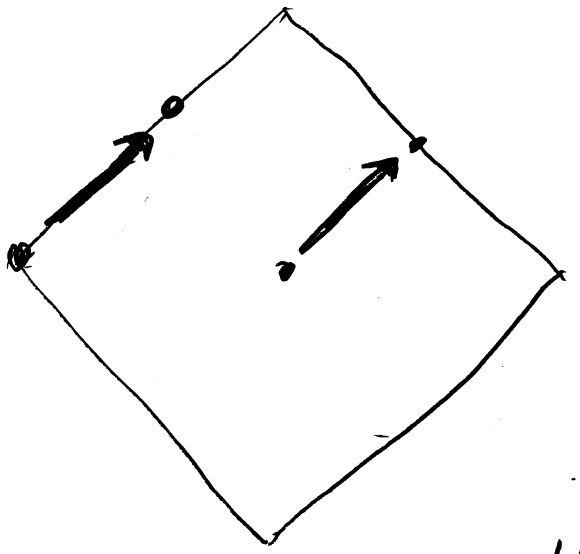
radial quantization

In other words, I think it is a "Hofman-Maldacena" map of radial quantization.

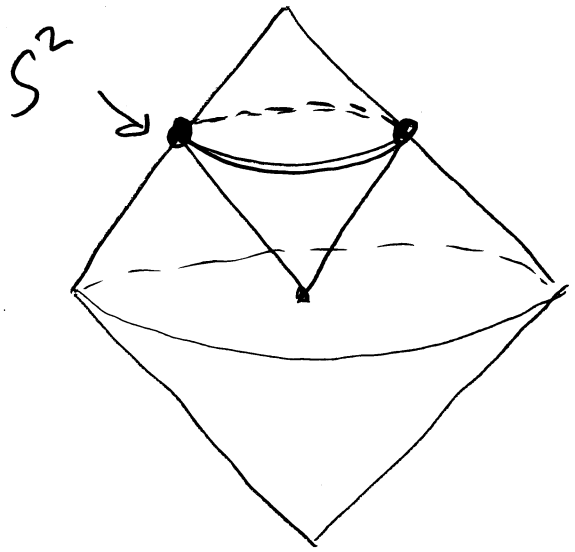
In lightcone coordinates $X^{\pm} = X^0 \pm X^3$

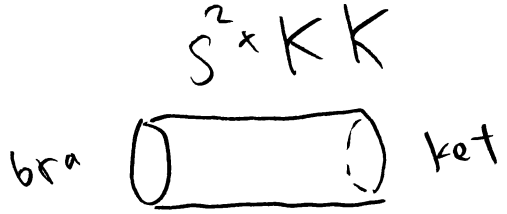
$$\Gamma: (X^+, X^-, X^i) = \left(-\frac{1}{X^+}, -\frac{X^2}{X^+}, -\frac{X^A}{X^+}\right)$$

Γ:



This map is complex in euclidean signature, hence "complex"





A CFT_4 restricted to a S^2 has CFT_2 symmetry

The $\delta_{\Delta_1}\delta_{\Delta_2}$ factor in RSW inner product

is then reminiscent of KK modes....

Thank You!