Dirichlet Gravitons in AdS_4 and the $\mathcal{L}_\Lambda w_{1+\infty}$ Wedge Algebra

Hare Krishna

Weinberg Institute, University of Texas at Austin

Based on arXiv:2609.36764

Motivation: soft gravitons with an AdS boundary

- **Flat spacetime.** Conformally soft gravitons generate the wedge of $w_{1+\infty}$.
- **AdS$_4$.** A curvature deformation, $\mathcal{L}_\Lambda w_{1+\infty}$, appears in self-dual gravity, twistor constructions and CFT$_3$ light-ray operators.
- **The role of boundary conditions.** Standard Dirichlet conditions couple the two graviton helicities. The bulk construction must incorporate this pairing.

The question

How do bulk gravitons obeying Dirichlet boundary conditions realize the deformed soft algebra?

We construct the bulk modes, determine their helicity pairing, and bootstrap their bracket from the AdS isometry action. We work with classical, linearized bulk modes.

Linearized gravitons on AdS_4

In conformally flat coordinates, the background metric is

$$d\bar{s}^2 = \frac{\eta_{\mu\nu}\, dx^\mu dx^\nu}{G(x)^2}, \qquad G(x) = 1 - \frac{x^2}{4\ell^2}, \qquad x^2 = \eta_{\mu\nu} x^\mu x^\nu.$$

Our curvature conventions are

$$\bar{R}_{\mu\nu} = \Lambda_{\rm Ein}\, \bar{g}_{\mu\nu}, \qquad \Lambda_{\rm Ein} = -\frac{3}{\ell^2}, \qquad \Lambda = \frac{\Lambda_{\rm Ein}}{3} = -\frac{1}{\ell^2}.$$

For $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, the linearized Einstein equation is

$$\mathcal{E}^{(1)}_{\mu\nu}[h] \equiv G^{(1)}_{\mu\nu}[h] + \Lambda_{\rm Ein}\, h_{\mu\nu} = 0.$$

The gauge equivalence is

$$h_{\mu\nu} \sim h_{\mu\nu} + 2\bar{\nabla}_{(\mu}\xi_{\nu)}.$$

We first organize solutions into the two helicity sectors. Dirichlet conditions will select their allowed combinations.

Spinor helicity basis in AdS

A null momenta can be written in terms of spinor helicity variable as

$$k_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}, \qquad k^2 = 0.$$

We initially treat λ and $\bar{\lambda}$ as independent complex spinors.

Little-group rescaling. The factorization has a redundancy:

$$(\lambda, \bar{\lambda}) \longmapsto (t\lambda, t^{-1}\bar{\lambda}), \qquad t \in \mathbb{C}^\times, \qquad k_{\alpha\dot{\alpha}} \longmapsto k_{\alpha\dot{\alpha}}.$$

Helicity

$$\hat{h} = \frac{1}{2}\left(\bar{\lambda}\cdot\partial_{\bar{\lambda}} - \lambda\cdot\partial_\lambda\right), \qquad \hat{h}\,\Psi_\sigma = \sigma\,\Psi_\sigma.$$

Then wavefunction satisfies

$$\Psi_\sigma(x; t\lambda, t^{-1}\bar{\lambda}) = t^{-2\sigma}\Psi_\sigma(x; \lambda, \bar{\lambda}), \qquad \sigma = \pm 2.$$

The AdS isometries in spinor variables

The complexified Lorentz algebra sits inside the AdS algebra:

$$\mathfrak{so}(3,1)_{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})_L \oplus \mathfrak{sl}_2(\mathbb{C})_R \subset \mathfrak{so}(3,2)_{\mathbb{C}}.$$

Six Lorentz generators:

$$\mathcal{M}_{\alpha\beta} = \lambda_\alpha \frac{\partial}{\partial \lambda^\beta} + \lambda_\beta \frac{\partial}{\partial \lambda^\alpha},$$

$$\bar{\mathcal{M}}_{\dot\alpha\dot\beta} = \bar\lambda_{\dot\alpha} \frac{\partial}{\partial \bar\lambda^{\dot\beta}} + \bar\lambda_{\dot\beta} \frac{\partial}{\partial \bar\lambda^{\dot\alpha}}.$$

Four AdS transvections:

$$\mathcal{P}_{\alpha\dot\alpha} = \lambda_\alpha \bar\lambda_{\dot\alpha} + \Lambda \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \bar\lambda^{\dot\alpha}}.$$

Their commutator closes on the Lorentz generators:

$$[\mathcal{P}, \mathcal{P}] \propto \Lambda\, (\mathcal{M} + \bar{\mathcal{M}}).$$

The derivative term makes AdS transvections noncommuting. As $\Lambda \to 0$, they reduce to multiplication by $k_{\alpha\dot\alpha}$.

Null directions, scale and Lorentz action

Separate the projective coordinates from the spinor scales:

$$\lambda_\alpha = a\, q_\alpha(z), \qquad q_\alpha(z) = (1, z),$$
$$\bar{\lambda}_{\dot\alpha} = b\, \bar{q}_{\dot\alpha}(\bar{z}), \qquad \bar{q}_{\dot\alpha}(\bar{z}) = (1, \bar{z}).$$

- $\omega = ab$ sets the scale of the null label and is invariant under little-group rescaling.
- $\rho = a/b$ carries the helicity dependence: $\rho\,\partial_\rho \Psi_\sigma = -\sigma \Psi_\sigma$.
- $(z, \bar{z})$ label null directions. They are independent in the complexified calculation. On the Lorentzian real slice, $\bar{z} = z^*$ gives an S^2.

The Lorentz generators act as

$$L_r = -z^{r+1}\partial_z + \frac{r+1}{2}\, z^r\big(\omega\partial_\omega - \sigma\big),$$
$$\bar{L}_r = -\bar{z}^{r+1}\partial_{\bar{z}} + \frac{r+1}{2}\, \bar{z}^r\big(\omega\partial_\omega + \sigma\big), \qquad r = -1, 0, 1.$$

The Mellin transform diagonalizes $\omega\partial_\omega$.

Mellin transform and Lorentz weights

The Mellin transform replaces the continuous spinor scale by a homogeneity label:

$$\widetilde{\Psi}_{\Delta,\sigma;\epsilon}(x;z,\bar{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1} e^{-\epsilon\omega} \Psi_\sigma(x;\omega,z,\bar{z}), \qquad \epsilon > 0.$$

Convergence and continuation

For the chosen AdS metric representative,

$$\Psi_\sigma = e^{i\omega Q\cdot x}\left(H^{(0)} + \frac{H^{(-1)}}{\omega}\right).$$

The regulator controls large ω. The curvature-dependent $1/\omega$ term requires $\mathrm{Re}\,\Delta > 1$ at small ω. Meromorphic continuation extends the family beyond this domain.

Integration by parts, followed by regulator removal in the distributional sense, gives

$$\omega\partial_\omega \longmapsto -\Delta, \qquad (\bar{h}, h) = \left(\frac{\Delta-\sigma}{2}, \frac{\Delta+\sigma}{2}\right), \quad \sigma = \pm 2.$$

Inverse Mellin transform and the physical contour

The regulated wavefunction is recovered by

$$e^{-\epsilon\omega}\Psi_\sigma(\omega) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\Delta\ \omega^{-\Delta}\widetilde{\Psi}_{\Delta,\sigma;\epsilon}.$$

Writing $\omega = e^u$ and $\Delta = c + i\nu$ turns this into Fourier inversion of $e^{cu-\epsilon e^u}\Psi_\sigma(e^u)$.

For ordinary inversion of these representatives,

$$c > 1 \quad \text{in AdS}, \qquad c > 0 \quad \text{in the flat limit.}$$

The principal-series contour

The Lorentz principal series has $\Delta = 1 + i\nu, \qquad \nu \in \mathbb{R}$.

- **Flat spacetime:** The physical inner product gives a delta-normalized basis on this contour.
- **AdS:** Lorentz unitarity of the Dirichlet graviton sector selects the same principal-series contour, $\Delta = 1 + i\nu$.

Mellin sectors and Lorentz action

Denote the continued positive-helicity Mellin sector by $\mathcal{H}^{p,+}_{\mu\nu}(x;z,\bar{z})$, with

$$\Delta = 4-2p, \qquad (\bar{h},h) = (1-p,\ 3-p).$$

The global Lorentz generators act as

$$\begin{aligned}\big[L_r,\mathcal{H}^{p,+}\big] &= \big[z^{r+1}\partial_z + (r+1)(3-p)z^r\big]\mathcal{H}^{p,+},\\ [\bar{L}_r,\mathcal{H}^{p,+}] &= \big[\bar{z}^{r+1}\partial_{\bar{z}} + (r+1)(1-p)\bar{z}^r\big]\mathcal{H}^{p,+},\end{aligned} \qquad r=-1,0,1.$$

Choose modes with definite Cartan eigenvalues:

$$[L_0, w^p_{\bar{m},m}] = -m\, w^p_{\bar{m},m}, \qquad [\bar{L}_0, w^p_{\bar{m},m}] = -\bar{m}\, w^p_{\bar{m},m}.$$

These eigenvalues determine the powers in the Laurent expansion.

Laurent modes as bulk solutions

Expand in the independent projective coordinates z and $\bar{z}$:

$$\mathcal{H}^{p,+}(x;z,\bar{z}) = \sum_{\bar{m},m} \bar{z}^{\,p-1-\bar{m}} z^{\,p-3-m}\, \frac{w^{p}_{\bar{m},m}(x)}{N_{p,\bar{m}}}.$$

The sum runs over all integers. The coefficients $N_{p,\bar{m}}$ are independent of x, with normalization left implicit for now.

Contour integrals extract each mode:

$$w^{p}_{\bar{m},m}(x) = N_{p,\bar{m}} \oint_{z=0} \frac{dz}{2\pi i}\, z^{m+2-p} \oint_{\bar{z}=0} \frac{d\bar{z}}{2\pi i}\, \bar{z}^{\bar{m}-p}\, \mathcal{H}^{p,+}(x;z,\bar{z}).$$

Bulk interpretation

The linearized Einstein operator acts only on x and commutes with mode extraction:

$$\mathcal{E}_x[\mathcal{H}^{p,+}] = 0 \quad \Longrightarrow \quad \mathcal{E}_x[w^{p}_{\bar{m},m}] = 0.$$

Each Laurent coefficient is therefore an on-shell bulk metric perturbation.

The full Laurent module

Laurent-completion assumption

The normalized bulk modes admit an AdS-equivariant extension to all integer Laurent powers, including negative powers:

$$p \in \tfrac{1}{2}\mathbb{Z}, \qquad p \pm \bar{m} \in \mathbb{Z}, \qquad p \pm m \in \mathbb{Z}.$$

We normalize these modes so that the global Lorentz action is

$$[\bar{L}_r, w^p_{\bar{m},m}] = \big[r(p-1) - \bar{m}\big]\, w^p_{\bar{m}+r,m},$$
$$[L_r, w^p_{\bar{m},m}] = \big[r(2-p) - m\big]\, w^p_{\bar{m},m+r}, \qquad r = -1, 0, 1.$$

- **Finite submodule.** For $p = 1, \frac{3}{2}, 2, \ldots$, the modes with $1 - p \le \bar{m} \le p - 1$ form a spin-$(p-1)$ multiplet of the barred $\mathfrak{sl}_2$ at fixed m. The barred raising and lowering coefficients vanish at the endpoints, closing this $(2p-1)$-dimensional multiplet.

The AdS wedge

Within the full Laurent module, we select the modes for which both Lorentz lowering chains terminate after finitely many steps.

Allowed modes

$$w^{p}_{\bar{m},m}, \qquad p \in \tfrac{1}{2}\mathbb{Z},$$

$$p \pm \bar{m} \in \mathbb{Z}, \qquad p \pm m \in \mathbb{Z},$$

$$\bar{m} \geq 1 - p, \qquad m \geq p - 2.$$

There are no upper bounds on $\bar{m}$ or m.

AdS transvections preserve this sector and connect neighbouring levels $p \to p \pm \frac{1}{2}$.

Dirichlet boundary condition

In Fefferman–Graham gauge, the transverse–traceless expansion about a flat boundary metric is

$$ds^2 = \frac{\ell^2}{r^2}\Big[dr^2 + (\eta_{ij} + \gamma_{ij})\, dy^i dy^j\Big],$$

$$\gamma_{ij} = \underbrace{s_{ij}}_{\delta g_{(0)ij}} + \frac{r^2}{2}\Box s_{ij} + r^3 \underbrace{t_{ij}}_{\propto\, \delta\langle T_{ij}\rangle} + O(r^4).$$

Dirichlet fixes $s_{ij} = 0$; t_{ij} determines the stress-tensor response.

The boundary Cotton–York tensor is

$$\mathrm{Cot}_{ij}[g_{(0)}] = \varepsilon_i{}^{kl}\nabla_k\left(R_{lj}[g_{(0)}] - \frac{1}{4}R[g_{(0)}]\, g_{(0)lj}\right).$$

Its linearization about η_{ij} in the transverse–traceless sector is

$$\mathscr{C}[s]_{ij} := \frac{d}{d\epsilon}\,\mathrm{Cot}_{ij}[\eta + \epsilon s]\Big|_{\epsilon=0} = -\frac{1}{2}\varepsilon_i{}^{kl}\partial_k\Box s_{lj}.$$

Dirichlet boundary conditions and helicity pairing

For the conformally rescaled metric, the finite Weyl data are

$$\mathcal{E}_{ij} := \lim_{r\to 0} r^{-1} E_{ij} = -\frac{3}{2}\, t_{ij}, \qquad \mathcal{B}_{ij} := \lim_{r\to 0} r^{-1} B_{ij} = -\mathscr{C}[s]_{ij}.$$

Thus Dirichlet boundary conditions imply $\mathcal{B}_{ij} = 0$.

Lorentzian chirality relates the electric and magnetic components:

$$\star C^{(\pm)} = \pm i C^{(\pm)} \qquad \Longrightarrow \qquad \mathcal{B}^{(\pm)} = \pm i \mathcal{E}^{(\pm)}.$$

Matching the electric data

$$\mathcal{B}_{\mathrm{tot}} = i\big(\mathcal{E}^{(+)} - \mathcal{E}^{(-)}\big) = 0 \quad \Longleftrightarrow \quad \mathcal{E}^{(+)} = \mathcal{E}^{(-)}.$$

The electric contributions add: $\mathcal{E}_{\mathrm{tot}} = 2\mathcal{E}^{(+)}$, so $t^{D}_{ij} = -\frac{4}{3}\mathcal{E}^{(+)}_{ij}$.

Vanishing magnetic data determine s_{ij} only modulo $\ker \mathscr{C}$. Dirichlet also fixes this remaining source to zero.

Lorentz covariance of the pairing

We seek a Lorentz-equivariant map between the helicity modules that matches their electric data:

$$\mathcal{E}_-\big(\mathcal{R}_D w\big) = \mathcal{E}_+(w).$$

A nonzero intertwiner must preserve the Lorentz Casimir difference:

$$\underbrace{4p-6}_{(\mathcal{C}_{\bar{L}}-\mathcal{C}_L)_+} = \underbrace{6-4q}_{(\mathcal{C}_{\bar{L}}-\mathcal{C}_L)_-} \qquad \Longrightarrow \qquad q = 3-p.$$

Cartan covariance preserves $(\bar{m}, m)$. On the chosen wedge, every mode lowers to $(\bar{m}, m) = (1-p, p-2)$; equivariance therefore leaves one coefficient ρ_p at each level.

Lorentz-covariant Dirichlet pairing

$$W^p_{\bar{m},m} = w^p_{\bar{m},m} + \mathcal{R}_D w^p_{\bar{m},m} = w^p_{\bar{m},m} + \rho_p\, \bar{w}^{3-p}_{\bar{m},m}.$$

The paired Mellin weights satisfy

$$\Delta_+ = 4-2p, \qquad \Delta_- = 2p-2, \qquad \Delta_+ + \Delta_- = 2.$$

AdS-covariant Dirichlet modes

The transvections connect neighboring levels; equivariance requires

$$\rho_{p-\frac{1}{2}} = \Lambda \rho_p, \qquad \rho_{p+\frac{1}{2}} = \Lambda^{-1} \rho_p.$$

At $p = \frac{3}{2}$, both Mellin weights equal one. Choose equal electric normalization, $\rho_{3/2} = 1$.

Dirichlet graviton modes

$$W^p_{\bar{m},m} = w^p_{\bar{m},m} + \Lambda^{3-2p}\, \bar{w}^{3-p}_{\bar{m},m} \qquad \Lambda = -\ell^{-2}.$$

The metric calculation verifies the electric-data matching:

$$\mathcal{E}_+(w^p_{\bar{m},m}) = \Lambda^{3-2p} \mathcal{E}_-(\bar{w}^{3-p}_{\bar{m},m}) \quad \Longrightarrow \quad s_{ij}[W^p_{\bar{m},m}] = 0.$$

The ten modes identified with the AdS generators are

$$\bar{L}_k = W^2_{k,0}, \qquad L_k = \Lambda^{-1} W^1_{0,k}, \qquad H_{\bar{r},r} = W^{3/2}_{\bar{r},r}.$$

$$k = -1, 0, 1, \qquad r, \bar{r} = \pm\tfrac{1}{2}.$$

Bootstrapping the soft bracket

We seek a bracket of the form

$$[W^p_{\bar{m},m}, W^q_{\bar{n},n}] = \sum_s C^s_{p,q}(\bar{m}, m; \bar{n}, n)\, W^s_{\bar{m}+\bar{n},m+n}.$$

Assumption: linear closure

The AdS isometry algebra and its classical charges close linearly. We assume the same structure for the soft modes, with field-independent coefficients and no central term.

Jacobi with the Cartan generators $L_0, \bar{L}_0$ fixes the output labels to $(\bar{m}+\bar{n}, m+n)$.

The two Lorentz lowering bounds, together with the mode lattice, restrict the output level:

$$p+q-2 \le s \le p+q-1, \qquad s-p-q \in \mathbb{Z} \quad \Longrightarrow \quad s = p+q-2 \text{ or } p+q-1.$$

Determining the structure constants

The ten global modes must reproduce the geometric AdS action on the entire wedge; in particular,

$$\left[H_{\bar{r},r}, W^{p}_{\bar{m},m}\right] = \left[\bar{r}(p-1) - \frac{\bar{m}}{2}\right] W^{p-\frac{1}{2}}_{\bar{m}+\bar{r},m+r} - \Lambda\left[r(p-2) + \frac{m}{2}\right] W^{p+\frac{1}{2}}_{\bar{m}+\bar{r},m+r}, \qquad r, \bar{r} = \pm\tfrac{1}{2}.$$

Jacobi with $L_{-1}, \bar{L}_{-1}$ determines the dependence on the mode labels up to functions of p, q.

Jacobi with a transvection relates these functions at neighboring levels:

$$[H,[X,Y]] = [[H,X],Y] + [X,[H,Y]], \qquad H = H_{\bar{r},r}.$$

Uniqueness within the linear ansatz

For $\Lambda \neq 0$, the transvection Jacobi identities link neighboring half-integer levels. Together with antisymmetry and the prescribed action of the ten global modes, these relations uniquely fix the remaining structure constants.

The Dirichlet soft algebra

The wedge of $\mathcal{L}_\Lambda w_{1+\infty}$ $\qquad (\Lambda = -\ell^{-2})$

$$\begin{aligned}\left[W^{p}_{\bar{m},m}, W^{q}_{\bar{n},n}\right] = &\left[(q-1)\bar{m} - (p-1)\bar{n}\right] W^{p+q-2}_{\bar{m}+\bar{n},m+n} \\ &- \Lambda\left[(q-2)m - (p-2)n\right] W^{p+q-1}_{\bar{m}+\bar{n},m+n}.\end{aligned}$$

At fixed algebra generators, the formal limit $\Lambda \to 0$ leaves the $w_{1+\infty}$ bracket in $(p, \bar{m})$, with m an additive loop label.

The Jacobi identity holds for arbitrary modes; the bracket closes within the wedge because coefficients vanish whenever an output would lie outside it.

Outlook: charges and the CFT_3 interpretation

1. **Relate the bulk and light-ray realizations.**

   Construct an explicit map between the Dirichlet graviton generators and the CFT_3 light-ray algebra of Strominger–Wei, starting with the ANEC modes and their conformal descendants.

2. **Understand the role of boundary conditions.**

   Compare with the higher-spin charges of Di Giacomo et al., constructed with boundary conditions allowing gravitational flux, and determine whether the Dirichlet mode algebra admits an integrable charge realization.

3. **Establish the operator dictionary in AdS/CFT.**

   Use the boundary stress-tensor response to identify the CFT observables associated with the bulk modes, determine their local or nonlocal character, and derive the corresponding Ward identities.

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Thank you!

Backup slides

An explicit graviton metric representative

On $\bar{g}_{\mu\nu} = G^{-2}\eta_{\mu\nu}$, take $k_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$ and define

$$a = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} x^{\alpha\dot{\alpha}} = -2k\cdot x, \qquad b = x_{\alpha\dot{\alpha}} x^{\alpha\dot{\alpha}} = -2x^2, \qquad G = 1 + \frac{b}{8\ell^2}. \tag{C.8}$$

Choose a reference spinor μ_α with $\langle \mu\lambda\rangle \neq 0$, and define

$$\langle \mu\lambda\rangle = \mu_\alpha \lambda^\alpha, \qquad \langle \mu x\lambda] = \mu^\alpha x_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}.$$

Positive-helicity potential

$$\begin{aligned} h^{(+)}_{\alpha\dot{\alpha},\beta\dot{\beta}} = &- \left(G - \frac{ib}{2\ell^2 a}\right) \frac{\mu_\alpha \mu_\beta \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\dot{\beta}}}{\langle \mu\lambda\rangle^2}\, e^{ik\cdot x} \\ &- \frac{ie^{ik\cdot x}}{2\ell^2 a} \frac{\mu_\alpha \mu_\beta \left(\lambda^\gamma x_{\gamma\dot{\alpha}} \bar{\lambda}_{\dot{\beta}} + \lambda^\gamma x_{\gamma\dot{\beta}} \bar{\lambda}_{\dot{\alpha}}\right)}{\langle \mu\lambda\rangle^3} \langle \mu x\lambda]. \end{aligned} \tag{C.9}$$

The displayed potential has local Lorentz-frame indices: $h_{\mu\nu} = \bar{e}^a{}_\mu \bar{e}^b{}_\nu h_{ab}$, $\bar{e}^a{}_\mu = G^{-1}\delta^a{}_\mu$.
The opposite helicity follows by exchanging dotted and undotted spinors.

Bootstrapping the bracket: allowed output levels

Inputs: linear closure, antisymmetry, covariance under the global seed, Jacobi.

$$[W^{p}_{\bar{m},m}, W^{q}_{\bar{n},n}] = \sum_{s} C^{s}\; W^{s}_{\bar{m}+\bar{n},\, m+n}$$

Distances from the lowest weight:
$a = \bar{m} + p - 1, \quad b = \bar{n} + q - 1, \quad c = m - p + 2, \quad d = n - q + 2.$

- $\bar{L}_{-1}$ is a derivation; it kills X after $a{+}1$ steps and Y after $b{+}1 \Rightarrow$ kills $[X, Y]$ after $a{+}b{+}1$ steps.
- Output distance $a_{\text{out}} = a + b + s - p - q + 1 \le a + b \;\Rightarrow\; s \le p + q - 1.$
- Same with L_{-1}: $\quad c_{\text{out}} = c + d + p + q - s - 2 \le c + d \;\Rightarrow\; s \ge p + q - 2.$
- Lattice: $s - p - q \in \mathbb{Z}$.

Two channels only

$$s \in \{\, p + q - 2,\;\; p + q - 1 \,\}$$

Flat-type channel $p + q - 2$ and curvature channel $p + q - 1$.

Covariance fixes the functional form

Shift to lowest-weight labels: $V^p_{a,c} \equiv W^p_{1-p+a,\,p-2+c}$

$$[V^p_{a,c}, V^q_{b,d}] = F_{p,q}(a,b;c,d)\, V^{p+q-2}_{a+b-1,\,c+d} + G_{p,q}(a,b;c,d)\, V^{p+q-1}_{a+b,\,c+d-1}$$

Leibniz rule for L_{-1}, $\bar{L}_{-1}$ gives recursions:

$$(c+d)F = c\,F(c{-}1,d) + d\,F(c,d{-}1), \qquad (a+b-1)F = a\,F(a{-}1,b) + b\,F(a,b{-}1)$$

$$(a+b)G = a\,G(a{-}1,b) + b\,G(a,b{-}1), \qquad (c+d-1)G = c\,G(c{-}1,d) + d\,G(c,d{-}1)$$

With antisymmetry:

$$F_{p,q} = \alpha_{p,q}\,a - \alpha_{q,p}\,b, \qquad G_{p,q} = \beta_{p,q}\,c - \beta_{q,p}\,d$$

Seeds from transvections on the corner $V^q_{0,0}$ ($V^{3/2}_{1,0} = H_{\frac{1}{2},-\frac{1}{2}}$, $V^{3/2}_{0,1} = H_{-\frac{1}{2},\frac{1}{2}}$):

$$[V^{3/2}_{1,0}, V^q_{0,0}] = (q-1)\,V^{q-\frac{1}{2}}_{0,0}, \qquad [V^{3/2}_{0,1}, V^q_{0,0}] = -\Lambda(q-2)\,V^{q+\frac{1}{2}}_{0,0}$$

$$\Rightarrow \quad \alpha_{3/2,q} = q-1, \qquad \beta_{3/2,q} = -\Lambda(q-2)$$

Jacobi with H_-: the constants are level-independent

$H_- \equiv H_{-\frac{1}{2},-\frac{1}{2}} = V_{0,0}^{3/2}$ acts as

$$[H_-, V_{a,c}^p] = -\frac{a}{2}\, V_{a-1,c}^{p-\frac{1}{2}} - \frac{\Lambda c}{2}\, V_{a,c-1}^{p+\frac{1}{2}}$$

Impose $[H_-,[X,Y]] = [[H_-,X],Y] + [X,[H_-,Y]]$, $X = V_{a,c}^p$, $Y = V_{b,d}^q$, level by level:

- **Lowest level** $p+q-\frac{5}{2}$ (no Λ):

  $$(a+b-1)\,F_{p,q}(a,b) = a\,F_{p-\frac{1}{2},q}(a-1,b) + b\,F_{p,q-\frac{1}{2}}(a,b-1) \quad \overset{(a,b)=(2,0)}{\Longrightarrow} \quad \alpha_{p,q} = \alpha_{p-\frac{1}{2},q}$$

- **Highest level** $p+q-\frac{1}{2}$ ($\propto \Lambda$):

  $$(c+d-1)\,G_{p,q}(c,d) = c\,G_{p+\frac{1}{2},q}(c-1,d) + d\,G_{p,q+\frac{1}{2}}(c,d-1) \quad \overset{(c,d)=(2,0)}{\Longrightarrow} \quad \beta_{p,q} = \beta_{p+\frac{1}{2},q}$$

- Middle levels mix F and G; they hold for the final bracket (next slide).

Chaining over $p \in \frac{1}{2}\mathbb{Z}$ from the seed $p = \frac{3}{2}$:

$\alpha_{p,q} = q-1$, $\beta_{p,q} = -\Lambda(q-2)$ $\Rightarrow$ $F = (q-1)a-(p-1)b$, $G = -\Lambda\big[(q-2)c-(p-2)d\big]$

Unique, given the inputs.

Jacobi for arbitrary modes: a Poisson realization

Monomial map:

$$\Phi\big(W^{p}_{\bar{m},m}\big) = x_1^{\,p-1+\bar{m}}\, x_2^{\,p-1-\bar{m}}\, y_1^{\,2-p+m}\, y_2^{\,2-p-m}$$

Constant Poisson bracket:

$$\{f,g\} = \tfrac{1}{2}\big(\partial_{x_1} f\, \partial_{x_2} g - \partial_{x_2} f\, \partial_{x_1} g\big) + \tfrac{\Lambda}{2}\big(\partial_{y_1} f\, \partial_{y_2} g - \partial_{y_2} f\, \partial_{y_1} g\big)$$

- x-plane gives $\bar{m}(q-1) - \bar{n}(p-1)$ and level $p+q-2$; y-plane gives $-\Lambda[m(q-2) - n(p-2)]$ and level $p+q-1$.
- So $\Phi([X,Y]) = \{\Phi(X), \Phi(Y)\}$, and Φ is injective (exponents fix $p, \bar{m}, m$).
- Constant bivector $\Rightarrow$ Jacobiator vanishes $\Rightarrow$ Jacobi holds for *all* modes.

Wedge

$a = p - 1 + \bar{m} \geq 0$, $c = 2 - p + m \geq 0$: polynomial in x_1, y_1 (Laurent in x_2, y_2). Closed, since $\partial_{x_1}, \partial_{y_1}$ only lower non-negative powers.

$\Lambda \to 0$: the y-plane drops out, leaving the area-preserving Poisson bracket behind $w_{1+\infty}$.

Stereographic coordinates and global AdS

The conformally flat coordinates used for the modes are stereographic:

$$ds^2 = G^{-2}\eta_{\mu\nu}dx^\mu dx^\nu, \qquad G = 1 - \frac{x^2}{4\ell^2}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-,+,+,+).$$

Embed AdS in $\mathbb{R}^{2,3}$ with $-(X^{-1})^2 - (X^0)^2 + \mathbf{X}^2 = -\ell^2$:

$$X^{-1} = \ell\,\frac{1 + x^2/(4\ell^2)}{G} = \ell\sec\rho\cos\tau, \qquad X^\mu = \frac{x^\mu}{G} = \ell\big(\sec\rho\sin\tau, \tan\rho\,\Omega\big).$$

Here $\Omega \in S^2$, $0 \le \rho < \pi/2$, and $\tau \in \mathbb{R}$ on the universal cover.

The coordinate transformation

$$\boxed{x^0 = \frac{2\ell\sin\tau}{\cos\tau + \cos\rho}, \qquad \mathbf{x} = \frac{2\ell\sin\rho}{\cos\tau + \cos\rho}\,\Omega} \qquad G = \frac{2\cos\rho}{\cos\tau + \cos\rho}.$$

$$ds^2 = \frac{\ell^2}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_2^2\right), \qquad ds^2_\partial \sim -d\tau^2 + d\Omega_2^2.$$

This chart uses $|\tau| < \pi - \rho$ and covers only part of global AdS.