Toward Celestial QED: Subleading Virtual Photons and Dressed Operators

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Motivation

In gauge theories and gravity, leading IR effects universally factorize and exponentiate:

$$\mathcal{A}(p_1,\ldots,p_N) = \underbrace{e^{S_L}}_{\text{soft factor}}\ \underbrace{H(p_1,\ldots,p_N)}_{\text{hard amplitude}}.$$

Leading soft/collinear factor
Universal and IR divergent
All-loop virtual photon or graviton exchange

Hard, IR-safe amplitude
Non-universal: theory and process dependent

In massless QED

$$\mathcal{A}_{\rm QED} = e^{-\alpha\sum_{n,m} Q_n Q_m \log|p_n\cdot p_m|} H(\{p_n\}), \qquad \alpha \equiv \frac{e^2}{4\pi^2}\log\frac{\Lambda_{\rm IR}}{\delta}.$$

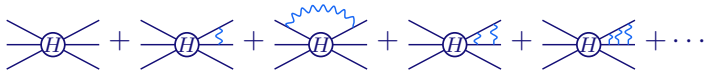
In gravity

$$\mathcal{A}_{\rm GRAV} = e^{-\gamma\sum_{n,m}(p_n\cdot p_m)\log(p_n\cdot p_m)} H, \qquad \gamma \equiv \frac{G_N}{\pi}\log\frac{\Lambda_{\rm IR}}{\delta}.$$

Here $\Lambda_{\rm IR}$ is the cutoff defining the soft/hard separation.

Motivation

Soft virtual photon corrections, at leading order, universally factorize and exponentiate.

$$\mathcal{A} = $$  $$+ \cdots$$

$$\mathcal{A} = H + S_1^{(0)} H + \frac{[S_1^{(0)}]^2}{2!} H + \frac{[S_1^{(0)}]^3}{3!} H + \cdots$$

$$= e^{S_1^{(0)}} H.$$

The one-loop $S_1^{(0)}$ iterates ("one-loop exactness").

$$S_1^{(0)} = -\alpha \sum_{n,m} Q_n Q_m \log \left| \frac{p_n \cdot p_m}{\mu^2} \right|, \qquad \alpha = \frac{e^2}{4\pi^2} \log \frac{\Lambda}{\delta}.$$

Λ: IR cutoff; δ: regulator.

[Yennie, Frautschi, Suura; 1961] [Weinberg; 1965]

$$\mathcal{A}_{\rm QED} = e^{-\alpha \sum_{n,m} Q_n Q_m \log|p_n\cdot p_m|} H \qquad \xrightarrow{\text{Mellin}}$$
$$\widetilde{\mathcal{A}} = e^{-\alpha \sum_{n,m} Q_n Q_m \log|z_{nm}|^2} \widetilde{H}(\{\Delta_n, z_n, \bar{z}_n\}).$$

$$e^{-\alpha \sum_{n,m} Q_n Q_m \log|z_{nm}|^2} = \prod_{n,m} (z_{nm}\bar{z}_{nm})^{-\alpha Q_n Q_m} = \langle V_{Q_1}(z_1)\cdots V_{Q_N}(z_N)\rangle.$$

[Arkani-Hamed, Pate, Raclariu, Strominger; 2020]

With local vertex operators

$$V_Q(z) \equiv :e^{iQ\phi(z)}:, \qquad S = \frac{1}{\alpha}\int \mathrm{d}^2x\,(\vec{\nabla}\phi)^2.$$

Local vertex operators + Gaussian field

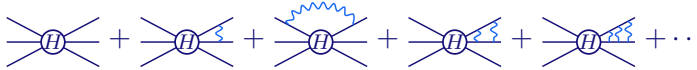
$\Longrightarrow$ full description of leading virtual soft divergences through all loops.

Can subleading contributions reveal more structure on the celestial sphere?

Can subleading contributions reveal more structure on the CS?

- We would first need some sort of universal factorization.
- It would also be nice to have an iteration such as the leading one.

$$\mathcal{A} = \quad + \quad + \quad + \quad + \quad + \cdots$$



$$H + S_1^{(0)} H + \frac{[S_1^{(0)}]^2}{2!} H + \frac{[S_1^{(0)}]^3}{3!} H + \cdots$$

We will show that, under certain circumstances, there is also a full iterative structure at subleading order.

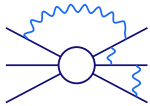
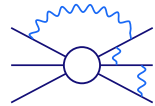
$$\mathcal{A} = \underbrace{e^{S_1^{(0)} + \epsilon S_1^{(1)}} (1 + \epsilon R^{(2)}) H}_{\text{still universal factorization}} .$$

$R^{(2)}$ is still process independent.

[Laenen, Stavenga, White; 2008]

[Laenen, Magnea, Stavenga, White; 2010] [Bonocore; 2020]

And when brought to the celestial sphere:

Leading soft exchange		Leading + subleading exchange
		
$V_Q = :e^{iQ\phi}:$	$\longrightarrow$	$V_Q\, e^{W_S}$
local operator		nonlocal dressing
$S = \frac{1}{\alpha}\int (\nabla\phi)^2$		$S = \frac{1}{\alpha}\int (\nabla\phi)^2 + \text{auxiliary fields}$
local action		local action

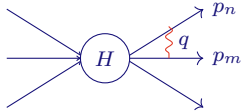
(*) See Choi and Mitra (2026): a Gaussian action reproduces leading factorization and IR divergences, and implements antipodal matching conditions and soft theorems.

(**) See also Nguyen, Rios, White (2023) for subleading effects from real soft photons on the dressing.

Review: leading iteration and exponentiation

Consider some amplitude; now we would like to compute

$\mathcal{A} =$  $+ \cdots$

One loop 

$$\mathcal{A}_1 = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \bar{w}_n(-ie_n\gamma^\mu)\frac{i(\not{p}_n + \not{q} + m)}{(p_n+q)^2 - m^2 + i0} \times H_0(p_n+q, p_m - q)\frac{i(-\not{p}_m + \not{q} + m)}{(p_m - q)^2 - m^2 + i0}(-ie_m\gamma^\nu)w_m \frac{-i\eta_{\mu\nu}}{q^2 + i0}.$$

Use the equations of motion $(\not{p} - m)u = 0$, $(\not{p} + m)v = 0$, and expand for soft q:

$$\mathcal{A}_{L,1} = -ie_n e_m (p_n \cdot p_m)\left[\int_{\text{soft}} \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(q^2+i0)(p_n\cdot q + i0)(-p_m \cdot q + i0)}\right] \underbrace{\bar{w}_n H_0 w_m}_{\mathcal{A}_0} \equiv S_L^{(1)}\mathcal{A}_0.$$

The integral

$$I(p_n, p_m) \equiv \int_{\text{soft}} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 + i0)(p_n \cdot q + i0)(-p_m \cdot q + i0)}, \qquad P_n^\mu = \eta_n p_n^\mu,$$

is usually evaluated using residues, yielding

$$I = -\frac{2\pi^2 i}{(p_n \cdot p_m)\beta_{nm}} \left[\log \frac{1+\beta_{nm}}{1-\beta_{nm}} - \underbrace{i\pi(1+\eta_n\eta_m)}_{\text{Coulomb phase}} \right] \log \frac{\Lambda_{\text{IR}}}{\delta},$$

$$\beta_{nm} = \sqrt{1 - \frac{m^4}{(p_n \cdot p_m)^2}}.$$

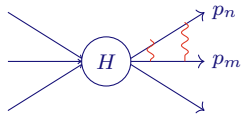
The phase is discarded in Weinberg's computation,
but we want to keep it.

In the massless limit,

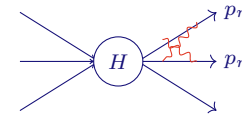
$$S_1^{(0)} \simeq \frac{e^2}{4\pi^2} \log \frac{\Lambda_{\text{IR}}}{\delta} \sum_{n,m} Q_n Q_m \log |p_n \cdot p_m|.$$

Note also that the Coulomb phase only appears for photon exchange among an incoming and an outgoing particle.

Two loops



planar



non-planar

For compactness, $D_n(k) = (p_n + k)^2 - m^2 + i0$ and $D_m(k) = (p_m - k)^2 - m^2 + i0$.

$$\mathcal{A}_P = \int \frac{\mathrm{d}^4 q_1 \mathrm{d}^4 q_2}{(2\pi)^8}\, \bar{w}_n(-ie_n\gamma^\mu)\frac{i(\not{p}_n + \not{q}_1 + m)}{D_n(q_1)}(-ie_n\gamma^\nu)\frac{i(\not{p}_n + \not{q}_1 + \not{q}_2 + m)}{D_n(q_1+q_2)}$$
$$\times H(p_n + q_1 + q_2, p_m - q_1 - q_2)\frac{i(-\not{p}_m + \not{q}_1 + \not{q}_2 + m)}{D_m(q_1+q_2)}$$
$$\times (-ie_m\gamma_\nu)\frac{i(-\not{p}_m + \not{q}_1 + m)}{D_m(q_1)}(-ie_m\gamma_\mu)w_m\frac{-i}{q_1^2 + i0}\frac{-i}{q_2^2 + i0}.$$

Add the non-planar contribution, keep leading powers in q_1, q_2, and symmetrize the integrands:

$$\int \mathrm{d}q_1 \mathrm{d}q_2\, [A_P(q_1,q_2) + A_{NP}(q_1,q_2)] \longrightarrow \frac{1}{2}[S_1^{(0)}]^2 \mathcal{A}_0.$$

Avoid overcounting under $q_1 \leftrightarrow q_2$.

And at three loops,

$$\mathcal{A}_{L,3} = \frac{[S_1^{(0)}]^3}{3!}\mathcal{A}_0.$$

$$\mathcal{A}_L = \mathcal{A}_0 + S_1^{(0)}\mathcal{A}_0 + \frac{[S_1^{(0)}]^2}{2!}\mathcal{A}_0 + \frac{[S_1^{(0)}]^3}{3!}\mathcal{A}_0 + \cdots$$

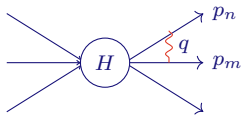
$$= e^{S_1^{(0)}}\mathcal{A}_0.$$

[Yennie, Frautschi, Suura; 1961] [Weinberg; 1965]

$$S_1^{(0)} = \frac{e^2}{4\pi^2}\log\frac{\Lambda}{\delta}\sum_{n,m} Q_n Q_m \log|p_n \cdot p_m|.$$

$$\therefore \quad \mathcal{A}_L \simeq e^{\alpha\sum_{n,m} Q_n Q_m \log|p_n\cdot p_m|}\mathcal{A}_0.$$

Subleading iteration and exponentiation



$$\mathcal{A}_1 = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \bar{w}_n(-ie_n\gamma^\mu)\frac{i(\not{p}_n + \not{q} + m)}{(p_n+q)^2 - m^2 + i0} H_0(p_n+q, p_m - q)$$
$$\times \frac{i(-\not{p}_m + \not{q} + m)}{(p_m - q)^2 - m^2 + i0}(-ie_m\gamma^\nu) w_m \frac{-i\eta_{\mu\nu}}{q^2 + i0}.$$

Now keep leading and subleading powers in q:

$$\frac{2p_n^\mu + \gamma^\mu \not{q}}{2p_n\cdot q + q^2} = \frac{p_n^\mu}{p_n\cdot q} + \Delta^\mu(p_n, q) - \frac{i\Sigma_n^{\mu\rho} q_\rho}{p_n\cdot q} + O(q),$$

$$\frac{-2p_m^\nu + \not{q}\gamma^\nu}{-2p_m\cdot q + q^2} = -\left[\frac{p_m^\nu}{p_m\cdot(-q)} + \Delta^\nu(p_m, -q) + \frac{i\Sigma_m^{\nu\rho}(-q_\rho)}{p_m\cdot(-q)}\right] + O(q).$$

$$\Delta^\mu(P,q) = \frac{(P\cdot q)q^\mu - q^2 P^\mu}{2(P\cdot q)^2}, \qquad \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

Define the current

$$J_n^\mu(q) \equiv Q_n \left[\frac{p_n^\mu}{p_n \cdot q} + \Delta^\mu(p_n, q) - \frac{i\Sigma_n^{\mu\rho} q_\rho}{p_n \cdot q} \right].$$

$$\mathcal{A}_1 = -i \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 + i0} \, \bar{w}_n \big[J_n(q) \cdot J_m(-q) \, H(p_n + q, p_m - q) \big]_{L+NL} w_m.$$

Expanding the subleading contributions from the hard amplitude,

$$H(p_n + q, p_m - q) = H(p_n, p_m) + \underbrace{q^\mu (\partial_{p_n^\mu} - \partial_{p_m^\mu}) H}_{\text{these will imprint bi-locality on the CS}} + O(q^2).$$

Absorb these derivatives into a redefinition of the current:

$$\Sigma_n^{\mu\rho} \quad \longrightarrow \quad \mathcal{J}_n^{\mu\rho}, \qquad J_n^\mu(q) = Q_n \left[\frac{p_n^\mu}{p_n \cdot q} + \Delta^\mu(p_n, q) - \frac{i\mathcal{J}_n^{\mu\rho} q_\rho}{p_n \cdot q} \right],$$

$$\mathcal{J}_n^{\mu\nu} = L_n^{\mu\nu} + \Sigma_n^{\mu\nu}, \qquad L_n^{\mu\nu} = -i \left(p_n^\mu \partial_{p_n^\nu} - p_n^\nu \partial_{p_n^\mu} \right).$$

$$\mathcal{A}_1 = -i\bar{w}_n \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[J_n^\mu(q) \frac{\eta_{\mu\nu}}{q^2 + i0} J_m^\nu(-q) \right]_{L+NL} H(p_n, p_m) w_m.$$

Observation:

$$J_n^\mu(q) = Q_n \left[\frac{p_n^\mu}{p_n \cdot q} + \frac{(p_n \cdot q) q^\mu - q^2 p_n^\mu}{2(p_n \cdot q)^2} - \frac{i \mathcal{J}_n^{\mu\rho} q_\rho}{p_n \cdot q} \right].$$

The q^2 term is absent for on-shell photons.

$$q_\mu J_n^\mu(q) = Q_n \left[1 + \underbrace{\frac{(p_n \cdot q) q^2 - q^2 (p_n \cdot q)}{2(p_n \cdot q)^2}}_{0} - \underbrace{\frac{i \mathcal{J}_n^{\mu\rho} q_\mu q_\rho}{p_n \cdot q}}_{0} \right] = Q_n.$$

$$\therefore \qquad q_\mu \sum_n J_n^\mu(q) = \sum_n Q_n = 0. \quad \text{Conserved!}$$

$$\mathcal{A}_1 = -i\bar{w}_n \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[J_n^\mu(q) \frac{\eta_{\mu\nu}}{q^2 + i0} J_m^\nu(-q) \right]_{L+NL} H(p_n, p_m) w_m.$$

Write $a = p_n \cdot q + i0$, $b = -p_m \cdot q + i0$, and $u = p_n \cdot p_m$. Then

$$\mathcal{A}_1 = -iQ_n Q_m \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 + i0} \bar{w}_n \Bigg[\underbrace{\frac{u H_0}{ab}}_{\text{leading}} - \underbrace{\frac{u q^2 (p_n - p_m)\cdot q}{2a^2 b^2} H_0}_{\text{subleading}}$$
$$+ \underbrace{\frac{\not{p}_m \not{q} H_0 - H_0 \not{q} \not{p}_n}{2ab}}_{\text{subleading}}$$
$$+ \underbrace{\frac{u\, q \cdot (\partial_{p_n} - \partial_{p_m}) H_0}{ab}}_{\text{subleading}} \Bigg] w_m.$$

- The leading term iterates at all loops: $[S_1^{(0)}]^\ell / \ell!$.
- Subleading terms also iterate, checked at two and three loops.
- A proof by induction would be nice (in progress).

$$\mathcal{A} = e^{W_L} e^{\epsilon W_S} (1 + \epsilon R^{(2)}) H, \qquad \epsilon \equiv \frac{\Lambda}{\mu}.$$

Or, through subleading order,

$$\mathcal{A} = e^{W_L} \left[1 + \epsilon (W_S + R^{(2)})\right] H.$$

$$W_L = \sum_{n<m} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} J^\mu_{n,L}(q) \frac{\eta_{\mu\nu}}{q^2 + i0} J^\nu_{m,L}(-q),$$

$$W_S = \sum_{n<m} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \Big[J^\mu_{n,S}(q) \frac{\eta_{\mu\nu}}{q^2 + i0} J^\nu_{m,L}(-q) + J^\mu_{n,L}(q) \frac{\eta_{\mu\nu}}{q^2 + i0} J^\nu_{m,S}(-q) \Big].$$

$$J_n = \underbrace{J_{n,L}}_{\text{leading}} + \underbrace{J_{n,S}}_{\text{subleading}} .$$

Dressed vertex operators on the celestial sphere

After Mellin transforming the full amplitude,

$$\widetilde{\mathcal{A}}(\{\Delta_n, z_n\}) = e^{i\Phi_C} \underbrace{e^{\alpha(\Lambda/\delta)\sum_{n<m} Q_n Q_m \log|z_{nm}|^2}}_{\langle V(z_1)\cdots V(z_N)\rangle} e^{(\Lambda/\mu)(W_1+W_2)} \widetilde{H}(\{\Delta'_n, z_n\}).$$

$$V(z_n) = :e^{iQ_n\phi(z_n)}:, \qquad \langle\phi(z)\phi(w)\rangle = -\alpha\log|z-w|^2.$$

Could a deformation of the leading vertex operators reproduce the subleading contributions? Recall, with $r_n = 1+|z_n|^2$,

$$W_1 = \sum_n \frac{1}{r_n} e^{-\partial_{\Delta_n}} \left\{ \partial_{\Delta_n} + \log\frac{2r_n\mu}{m} \right\},$$

$$W_2 = -\sum_{\substack{n\neq m \\ \eta_n\eta_m=-1}} Q_n Q_m \log\frac{r_n r_m}{|z_{nm}|^2}\, e^{-\partial_{\Delta_n}} \left[\frac{z_{nm}\partial_{z_n} + 2h_n}{1+\bar{z}_n z_m} + \frac{\bar{z}_{nm}\partial_{\bar{z}_n} + 2\bar{h}_n}{1+z_n\bar{z}_m} \right].$$

Observation: W_2 correlates incoming with outgoing states only.

The answer seems to be positive. On top of the free field $\phi(z)$, introduce an auxiliary field $\chi_\eta(z)$ that detects the charge of another vertex when contracted with it.

$$\begin{aligned}
\langle \phi_\eta(z)\phi_{\eta'}(w)\rangle &= -\alpha \log|z-w|^2, \qquad && \eta = \pm 1 \quad \text{(in/out)},\\
\langle \chi_\eta(z)\phi_{\eta'}(w)\rangle &= -i\delta_{\eta\eta'}\delta^{(2)}(z-w), && \\
\langle \chi_\eta(z)\chi_{\eta'}(w)\rangle &= 0, && \alpha = \frac{e^2}{4\pi^2}\log\frac{\Lambda}{\delta}.
\end{aligned}$$

The dressed insertion would then be, with $r(z) = 1 + |z|^2$,

$$\begin{aligned}
\mathbb{V}_{Q,\eta}(z) = \ :e^{iQ\phi_\eta(z)}\Bigg\{ & 1 + \frac{\Lambda}{\mu}\frac{1}{r(z)}\left[\partial_\Delta + \log\frac{2\mu r(z)}{m}\right]e^{-\partial_\Delta} \\
& - \frac{\Lambda}{\mu}\frac{Q}{r(z)}e^{-\partial_\Delta}\int \mathrm{d}\Omega_w\, \chi_\eta(w) \\
& - \frac{\Lambda}{\mu}Q\int \mathrm{d}\Omega_w\, \log\frac{r(z)r(w)}{|z-w|^2}\,\chi_{-\eta}(w)\,\mathcal{K}_{h,\bar h}(z,w)\Bigg\}:\ .
\end{aligned}$$

with the kernel

$$\mathcal{K}_{h,\bar h}(z,w) = e^{-\partial_\Delta}\left[\frac{(z-w)\partial_z + 2h}{1+\bar z w} + \frac{(\bar z - \bar w)\partial_{\bar z} + 2\bar h}{1 + z\bar w}\right].$$

$$\mathbb{V}_{Q,\eta}(z) = \,:e^{iQ\phi_\eta(z)}\Bigg\{1+\frac{\Lambda}{\mu}\frac{1}{r(z)}\left[\partial_\Delta+\log\frac{2\mu r(z)}{m}\right]e^{-\partial_\Delta}$$
$$-\frac{\Lambda}{\mu}\frac{Q}{r(z)}e^{-\partial_\Delta}\int \mathrm{d}\Omega_w\,\chi_\eta(w)$$
$$-\frac{\Lambda}{\mu}Q\int \mathrm{d}\Omega_w\,\log\frac{r(z)r(w)}{|z-w|^2}\,\chi_{-\eta}(w)\,\mathcal{K}_{h,\bar{h}}(z,w)\Bigg\}:\,.$$

Observations:

- A single local vertex $V_Q = \,:e^{iQ\phi}:$ captures the full leading effect of the external cloud of virtual photons. The subleading contributions introduce a nonlocal description on the celestial sphere.

- The leading soft vertex is insensitive to the helicities of the hard particles. The subleading dressing contains angular momentum operators that do see the hard-particle spin, just as in real soft emissions.

Outlook

- A different exponentiation, but a complete one.

[Grammer, Yennie; 1973]

$$\frac{\eta_{\mu\nu}}{q^2+i0} = \underbrace{\frac{G^{(ij)}_{\mu\nu}}{q^2+i0}}_{G\text{-photons}} + \underbrace{\frac{K^{(ij)}_{\mu\nu}}{q^2+i0}}_{K\text{-photons}},$$

$$G^{(ij)}_{\mu\nu} = \eta_{\mu\nu} - b^{(ij)} q_\mu q_\nu, \qquad K^{(ij)}_{\mu\nu} = b^{(ij)} q_\mu q_\nu.$$

$$\mathcal{A} = \exp\left\{ \sum_{i,j} \int \mathrm{d}^4 q_k \, \frac{b_{ij}(p_i, p_j; q_k)}{q_k^2} \right\} \mathcal{A}_G.$$

$\mathcal{A}_G$ is not the usual hard amplitude, but is also IR safe.

Including the subleading contribution,

$$\mathcal{A} = \exp\left\{ \sum_{i,j} \int \mathrm{d}^4 q_k \, \frac{b_{ij}(p_i, p_j; q_k)}{q_k^2} \left[1 + q_k \cdot (\partial_{p_i} + \partial_{p_j})\right] \right\} \mathcal{A}_G.$$

- Extend to subleading soft virtual gravitons.
- Study the properties of the dressed vertex operators.

Thank you!