Logarithmic soft photon theorem and waveform tails in higher dimensions

No infrared divergence, yet universal soft logarithms?
(Long-range physics beyond four dimensions)

Based on arXiv:2608.03747 [hep-th]

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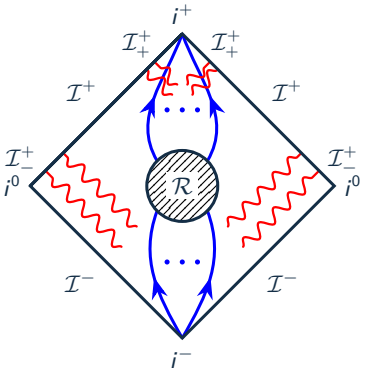


Celestial Holography Workshop @ Harvard University

Part I

Introduction to the problem

What part of electromagnetic waveform can be universal?



Scattering data

Outgoing: $q_a, m_a, p_a = m_a v_a$.

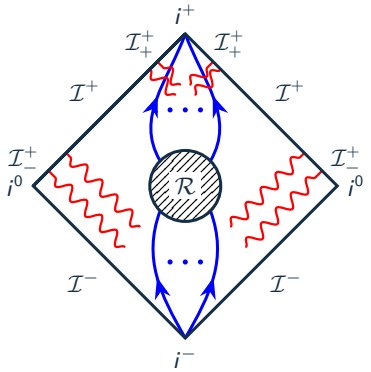
Incoming: $q'_a, m'_a, p'_a = m'_a v'_a$.

L: size and duration of the interaction region $\mathcal{R}$.

Radiative waveform at $\vec{x} = r\hat{x}$ and $t = u + r$:

$$A^\mu(t, r\hat{x}) \simeq \frac{A^\mu_{\rm rad}(u, \hat{x})}{r^{(d-2)/2}}.$$

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Observation: For $|u| \gg L$, the radiative waveform $A^\mu_{\rm rad}(u, \hat{x})$ is generated by the asymptotic motion of the charged particles.

Expectation: The universal long-range electromagnetic force controls the asymptotic trajectories, so the large-$|u|$ behaviour of $A^\mu_{\rm rad}(u, \hat{x})$ should depend only on the scattering data.

Late and early waveforms probe nonanalytic frequency dependence

Fourier-transform the radiative mode directly in retarded time:

$$\widetilde{A}^{\mu}_{\rm rad}(\omega,\hat{x}) = \int_{-\infty}^{\infty} \mathrm{d}u\, e^{i\omega u} A^{\mu}_{\rm rad}(u,\hat{x}),$$

$$A^{\mu}_{\rm rad}(u,\hat{x}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, e^{-i\omega u} \widetilde{A}^{\mu}_{\rm rad}(\omega,\hat{x}).$$

A frequency power series

$$\omega^j \quad \longleftrightarrow \quad i^j \delta^{(j)}(u),$$

$$j \in \mathbb{Z}_{\geq 0}.$$

Each term is localized near $u \sim 0$; does not contribute to $A^{\mu}_{\rm rad}(u,\hat{x})$ for $|u| \gg L$.

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A nonanalytic frequency term

$$\omega^j \ln(\omega \pm i0) \quad \longleftrightarrow \quad \frac{\Theta(\pm u)}{|u|^{j+1}}, \qquad j \in \mathbb{Z}_{\geq 0}.$$

Generates late and early time tails.

To find power-law tails at large $|u|$, study the nonanalytic terms as $\omega \to 0$.

Relation between radiative waveform and soft-photon factor

$$\varepsilon_\mu \widetilde{A}^\mu_{\rm rad}(\omega, \hat{x}) \simeq \frac{(\omega + i0)^{(d-4)/2}}{2(2\pi i)^{(d-2)/2}} \, S^{\rm cl}_{\rm em}$$

$$\mathcal{M}_{n+1} = S_{\rm em} \, \mathcal{M}_n + \text{non-factorized terms}$$

[Laddha and Sen, 2018, 2019]

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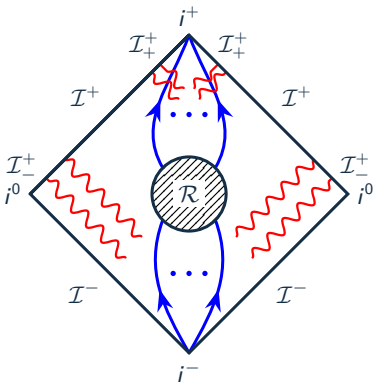
$d>4$	$A^\mu_{\rm rad}(u,\hat{x})$	$\widetilde{A}^\mu_{\rm rad}(\omega,\hat{x})$	$S^{\rm cl}_{\rm em}$
Even	$\partial_u^{(d-4)/2}\Theta(u)$	$\omega^{(d-6)/2}$	ω^{-1}
	$\partial_u^{p+(d-4)/2}\delta(u)$	$\omega^{p+(d-4)/2}$	ω^{p}
	$u^{-(3d-10)/2}$	$\omega^{3(d-4)/2}\ln\omega$	$\omega^{d-4}\ln\omega$
Odd	$u^{-(d-4)/2}$	$\omega^{(d-6)/2}$	ω^{-1}
	$u^{-p-(d-2)/2}$	$\omega^{p+(d-4)/2}$	ω^{p}
	$u^{-(3d-10)/2}\ln u$	$\omega^{3(d-4)/2}\ln\omega$	$\omega^{d-4}\ln\omega$

The acceleration tails indicate presence of logarithmic soft factor despite IR finiteness.

Part II

Classical waveform tails

Electromagnetic waveform observation window



Scales and observation direction

L: size and duration of $\mathcal{R}$, $\vec{x} = r\hat{x}$: detector position
$u = (t - r)$: retarded time, ω: frequency of waveform

$$\mathbf{n}^\mu = (1, \hat{x}), \qquad k^\mu = \omega \mathbf{n}^\mu.$$

$L \ll |u|$: resolve the asymptotic motion
Hard-region corrections are suppressed by $L/|u|$.

$|u| \ll r$: stay in the radiation zone
Far-field corrections are suppressed by $|u|/r$.

$$r^{-1} \ll |\omega| \sim |u|^{-1} \ll L^{-1}$$

Take large r before large $|u|$: extract the $r^{-(d-2)/2}$ radiative waveform as large u expansion.

Even $d \geq 6$: strategy for deriving late-time waveform

1. Solve Maxwell's equation with retarded Greens function:

In Lorenz gauge $\partial_\mu A^\mu = 0, \qquad \Box A^\mu = -J^\mu,$

$$\boxed{A^\mu(x) = -\int \mathrm{d}^d y \; G_R(x-y) J^\mu(y)}$$

$$G_R(x) = -\frac{\Theta(x^0)}{2\pi^{(d-2)/2}} \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{(d-4)/2} \delta(s)\Big|_{s=(x^0)^2-|\vec{x}|^2} .$$

Θ: step function; $\delta(s)$: one-dimensional Dirac delta.

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2. Determine the trajectories that supply the outgoing current:

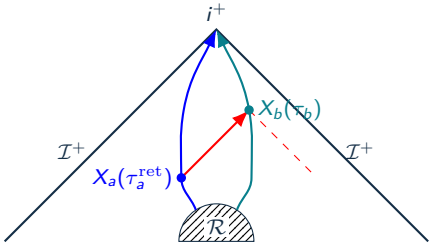
$$J^\mu_{\mathrm{out}}(y) = \sum_b q_b \int_0^\infty \mathrm{d}\tau_b \, \dot{X}^\mu_b(\tau_b) \, \delta^{(d)}\big(y - X_b(\tau_b)\big).$$

Find X_b from its equation of motion.

3. Insert the current in the boxed equation and take the radiative limit:

At $x^\mu = (u + r, r\hat{x})$, retain order $r^{-(d-2)/2}$ in $L \ll u \ll r$.

Correction to straight-line trajectory



Past light cone of X_b:
retarded field sourced by particle a
accelerates particle b.

$$m_b \ddot{X}_b^\mu = q_b \sum_{a \neq b} F_{(a)}^{\mu\nu}(X_b) \dot{X}_{b\nu}.$$

$$\ddot{X}_b^\mu \simeq \frac{(d-3)c_b^\mu}{\tau_b^{d-2}}, \qquad \dot{X}_b^\mu \simeq v_b^\mu - \frac{c_b^\mu}{\tau_b^{d-3}}.$$

$$\boxed{X_b^\mu \simeq r_b^\mu + v_b^\mu \tau_b + \frac{c_b^\mu}{d-4} \tau_b^{-(d-4)}}$$

The coefficient c_b^μ is fixed entirely by outgoing scattering data:

$$c_b^\mu \equiv \frac{q_b}{m_b} \frac{\Gamma(d-3)}{(4\pi)^{(d-2)/2}\Gamma((d-2)/2)} \sum_{a \neq b} q_a \frac{v_a^\mu v_b^2 - (v_a \cdot v_b) v_b^\mu}{[(v_a \cdot v_b)^2 - v_a^2 v_b^2]^{(d-1)/2}}.$$

Consistency check: in $d \to 4$ limit, the trajectory correction generates $\ln \tau_b$.

From outgoing trajectories to the late-time waveform

Localize the retarded field on the detector's past light cone:
For $L \ll u \ll r$, its null constraint simplifies to

$$0 = (x^0 - X_b^0)^2 - |\vec{x} - \vec{X}_b|^2 = 2r(u + \mathbf{n} \cdot X_b) + O(u^2), \qquad \tau_b^{\rm ret} \simeq -\frac{u}{\mathbf{n} \cdot v_b}.$$

$$\boxed{A^\mu_{\rm out} \simeq -\frac{1}{2(2\pi)^{(d-2)/2} r^{(d-2)/2}} \left(\frac{\rm d}{{\rm d}u}\right)^{(d-4)/2} \sum_b q_b \left.\frac{\dot{X}_b^\mu}{\mathbf{n} \cdot \dot{X}_b}\right|_{\tau_b^{\rm ret}}}$$

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Substituting trajectory result from the preceding slide one gets waveform tail:

$$\left[\frac{\dot{X}_b^\mu}{\mathbf{n} \cdot \dot{X}_b}\right]_{\rm acc} \sim u^{-(d-3)} \quad \Longrightarrow \quad \boxed{A^\mu_{\rm acc,out} \sim r^{-(d-2)/2} u^{-(3d-10)/2}}$$

Remark: For even $d \geq 6$, the constant-velocity waveform is the Low–Weinberg coefficient times $\left(\frac{\rm d}{{\rm d}u}\right)^{(d-4)/2} \Theta(u - \mathcal{O}(L))$, up to the radiative prefactor. Does not survive when $u >> L$. No electromagnetic memory at leading radiative order.

[Garfinkle, Hollands, Ishibashi, Tolish and Wald, 2017; Laddha and Sen, 2019]

Universal early- and late-time tails in even $d \geq 6$

$$A^{\mu}_{\rm acc} \simeq \frac{\Gamma((3d-10)/2)}{(2\pi)^{(d-4)/2}(4\pi)^{d/2}\Gamma((d-2)/2)}\frac{1}{r^{(d-2)/2}}\begin{cases}\dfrac{(-1)^{(d-2)/2}\mathcal{C}^{\mu}_{\rm out,even}}{u^{(3d-10)/2}}, & u>0,\\[2ex] -\dfrac{\mathcal{C}^{\mu}_{\rm in,even}}{|u|^{(3d-10)/2}}, & u<0.\end{cases}$$

with

$$\mathcal{C}^{\mu}_{\rm out,even}(\mathbf{n}) = \sum_b \sum_{a\neq b} q_b^2 q_a \frac{m_b^2 m_a^{d-2}(p_b\cdot\mathbf{n})^{d-5}}{[(p_a\cdot p_b)^2 - m_a^2 m_b^2]^{(d-1)/2}} \times \left[(p_a\cdot\mathbf{n})p_b^{\mu} - (p_b\cdot\mathbf{n})p_a^{\mu}\right].$$

out: outgoing pairs; in: replace by incoming (primed) data.

Universality: Charges and asymptotic momenta fix the leading waveform tail.

Even versus odd dimensions: retarded propagation

The odd-dimensional retarded Green's function:

$$G_R(x) = \frac{(-1)^{(d-1)/2}\Gamma((d-2)/2)}{2\pi^{d/2}}\Theta(x^0)\ \mathrm{Fp}\left[\Theta(s)s^{-(d-2)/2}\right], \qquad s = (x^0)^2 - |\vec{x}|^2.$$

Fp: finite-part distribution at the light cone $s = 0$.

	Even $d \geq 6$	**Odd** $d \geq 5$
Retarded support	On the light cone	On and inside the light cone
Late outgoing force	Outgoing sources only	Outgoing and incoming sources
Early incoming force	Incoming sources only	Incoming sources only
Velocity correction	$\dot{Y} \sim \lvert\tau\rvert^{-(d-3)}$	Same power; new coefficients

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Straight-line tail, $u > 0$	No tail for $u \gg L$	$u^{-(d-4)/2}$ propagation tail [Satishchandran and Wald, 2017; Laddha and Sen, 2019]
Acceleration tail, $u > 0$	$u^{-(3d-10)/2}$	$(\ln u)\, u^{-(3d-10)/2}$
Acceleration tail, $u < 0$	$\lvert u\rvert^{-(3d-10)/2}$	$\lvert u\rvert^{-(3d-10)/2}$; no $\ln\lvert u\rvert$

Propagation inside the light cone adds a late-time logarithm to the acceleration tail.

Universal early- and late-time tails in odd $d \geq 5$

For $\alpha = (3d-10)/2$ in the region $L \ll |u| \ll r$:

$$A^{\mu}_{\rm acc} \simeq \frac{\Gamma(\alpha)}{4\pi^2(2\pi)^{3(d-2)/2} r^{(d-2)/2}} \begin{cases} -\dfrac{\ln u}{u^{\alpha}} \left(\mathcal{C}^{\mu}_{\rm out,odd} + \mathcal{C}^{\mu}_{\rm mix,odd} + \mathcal{C}^{\mu}_{\rm in,odd} \right), & u > 0, \\ \dfrac{(-1)^{(d-3)/2}\pi}{|u|^{\alpha}} \mathcal{C}^{\mu}_{\rm in,odd}, & u < 0. \end{cases}$$

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Scattering-data coefficients (a: source; b: accelerated particle):

$$\mathcal{C}^{\mu}_{\rm out,odd} = \sum_b \sum_{a \neq b} \frac{q_b^2 q_a (-p_b \cdot \mathbf{n})^{d-4}}{m_a m_b^{d-3}} \mathcal{F}^{(d)}_{\rm out}\left(\frac{p_a \cdot p_b}{m_a m_b} \right) \left(p_a^{\mu} - \frac{p_a \cdot \mathbf{n}}{p_b \cdot \mathbf{n}} p_b^{\mu} \right),$$

$$\mathcal{C}^{\mu}_{\rm in,odd} = \mathcal{C}^{\mu}_{\rm out,odd}\Big|_{(q,p,m)\to(q',p',m'),\ \mathcal{F}_{\rm out}\to\mathcal{F}_{\rm in}},$$

$$\mathcal{C}^{\mu}_{\rm mix,odd} = \sum_{b\in{\rm out}} \sum_{a\in{\rm in}} \frac{q_b^2 q_a' (-p_b \cdot \mathbf{n})^{d-4}}{m_a' m_b^{d-3}} \mathcal{F}^{(d)}_{\rm mix}\left(-\frac{p_a' \cdot p_b}{m_a' m_b} \right) \left(p_a'^{\mu} - \frac{p_a' \cdot \mathbf{n}}{p_b \cdot \mathbf{n}} p_b^{\mu} \right).$$

$\mathcal{F}^{(d)}_{\cdots}$: dimensionless scalar functions of relative velocity.

The late logarithm combines all three histories; the early tail depends only on incoming data.

Part III

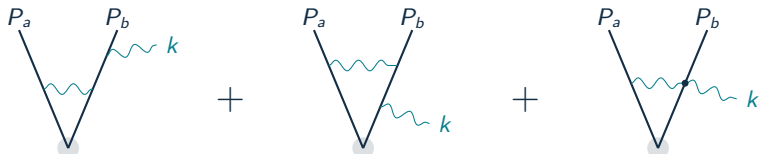
Logarithmic soft factors from the S-matrix

From one-loop diagrams to the logarithmic soft factor

Theory: Massive scalar QED with a non-derivative, gauge-invariant hard interaction, represented by blob.

$$(P_a, Q_a, \eta_a) = \begin{cases} (p_a, q_a, +1), & \text{out}, \\ (-p'_a, -q'_a, -1), & \text{in}, \end{cases} \qquad P_a^2 = -\mathfrak{m}_a^2, \qquad n: \text{ total charged lines.}$$

$\mathcal{M}^{(1)}_{n+1}$ relevant part $=$



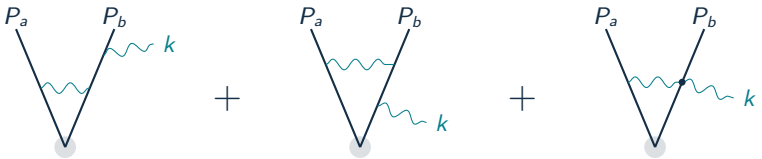
1. Sum the three integrands and then subtract $S^{(-1)}\mathcal{M}^{(1)}_n$, where $S^{(-1)} = \sum_a Q_a\, \varepsilon \cdot P_a/(k \cdot P_a)$.

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$\mathcal{M}_{n+1}^{(1)} \overset{\text{relevant part}}{=}$



1. Sum the three integrands and then subtract $S^{(-1)}\mathcal{M}_n^{(1)}$, where $S^{(-1)} = \sum_a Q_a\, \varepsilon \cdot P_a/(k \cdot P_a)$.

2. Isolate the marginal term: Expand in k/ℓ for $\omega \ll |\ell| \ll \Lambda$ (ℓ: loop momentum; Λ: hard scale):

$$\omega^j \int_\omega^\Lambda \mathrm{d}|\ell|\, |\ell|^{d-5-j} \quad \xrightarrow{j=d-4} \quad \omega^{d-4} \int_\omega^\Lambda \frac{\mathrm{d}|\ell|}{|\ell|}.$$

3. Evaluate the logarithmic loop integral by expressing it as a momentum differential on a scalar Feynman integral.

A marginal loop-momentum region generates the logarithm despite IR finiteness.

Integral expression of logarithmic soft factor

$$\left[\mathcal{M}_{n+1}^{(1)} - S^{(-1)}\mathcal{M}_n^{(1)}\right]_{\omega^{d-4}\ln\omega} = S_{\log}\,\mathcal{M}_n^{(0)}.$$

$$S_{\log} = \frac{i}{d-3}\sum_{a\neq b} Q_a Q_b^2 (k\cdot P_b)^{d-4}\left(\varepsilon_\rho - \frac{\varepsilon\cdot P_b}{k\cdot P_b}k_\rho\right)$$
$$\times\,(P_a^\rho P_{b\sigma} - (P_a\cdot P_b)\delta^\rho{}_\sigma)\,\frac{\partial\,\mathcal{K}_{ab}^{\mathrm{F}}\big|_{\ln\omega}}{\partial P_{b\sigma}},$$

$$\mathcal{K}_{ab}^{\mathrm{F}} = \int_\omega^\Lambda \frac{\mathrm{d}^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2 - i0)(P_a\cdot\ell - i0)(P_b\cdot\ell + i0)^{d-3}}$$

For $d = 4$, $S_{\log}$ can be re-written as the tree-level subleading soft operator operating on the regulated eikonal factor; this relation does not hold for $d > 4$.

$d = 4$: [**BS** and Sen, 2018]

The one-loop logarithmic soft-photon theorem

Evaluating $\mathcal{K}^{\mathrm{F}}_{ab}$ and differentiating gives

$$\boxed{S_{\log} = \ln\omega \sum_{a\neq b} Q_a Q_b^2 (k\cdot P_b)^{d-4}\left(\varepsilon\cdot P_a - \frac{k\cdot P_a}{k\cdot P_b}\varepsilon\cdot P_b\right) F^{(d)}_{ab}}$$

The evaluated pair coefficient (both even and odd $d > 4$):

$$F^{(d)}_{ab} = -\frac{i(1+\eta_a\eta_b)}{2}\,\mathcal{C}_d\,\frac{\mathrm{m}_a^{d-2}\mathrm{m}_b^2}{\Delta_{ab}^{(d-1)/2}} - \frac{(-1)^d\eta_a\eta_b}{(d-3)2^{d-1}\pi^{d/2}\Gamma((d-2)/2)}\,\frac{(d-3)\mathcal{H}_d(s_{ab}) + s_{ab}\mathcal{H}'_d(s_{ab})}{\mathrm{m}_a\mathrm{m}_b^{d-3}},$$

$$\Delta_{ab} = (P_a\cdot P_b)^2 - \mathrm{m}_a^2\mathrm{m}_b^2, \qquad s_{ab} = -\eta_a\eta_b\frac{P_a\cdot P_b}{\mathrm{m}_a\mathrm{m}_b} > 1.$$

$\mathcal{C}_d$: dimension-dependent normalization; $\mathcal{H}_d$: scalar function of relative velocity; $\mathcal{H}'_d = \mathrm{d}\mathcal{H}_d/\mathrm{d}s$.

Part IV

Summary and Outlook

Summary: soft orders and late-time waveform tails

$$\mathcal{M}_{n+1} = S_{\rm em}\,\mathcal{M}_n + \text{non-factorized terms},$$

$$S_{\rm em} \sim \underbrace{\frac{1}{\omega}}_{S^{(-1)}} + \underbrace{\sum_{p\geq 0}\mathcal{O}(\omega^p)}_{\text{analytic, partially factorized for } p\geq 1} + \underbrace{\omega^{d-4}\ln\omega}_{S_{\log}} + \cdots .$$

Late time leading radiative order waveform:

$$A^{\mu}_{\rm even}(u,r) \sim \frac{1}{r^{(d-2)/2}}\left[\underbrace{\partial_u^{(d-4)/2}\Theta(u)}_{u\gg L:\to\ 0} + \underbrace{\sum_{p\geq 0}\partial_u^{p+(d-4)/2}\delta(u)}_{u\gg L:\to\ 0} + \frac{1}{u^{(3d-10)/2}} + \cdots\right]$$

$$A^{\mu}_{\rm odd}(u,r) \sim \frac{1}{r^{(d-2)/2}}\left[\frac{1}{u^{(d-4)/2}} + \sum_{p\geq 0}\frac{1}{u^{p+(d-2)/2}} + \frac{\ln u}{u^{(3d-10)/2}} + \cdots\right]$$

Outlook: three open directions

1. Quantum universality and extension to all soft orders

Establish universality beyond scalar QED, test one-loop exactness, and extend the logarithmic expansion to all soft orders, guided by the four-dimensional results.

$d = 4$: [Krishna and **BS**, 2023; Karan, Khatun, **BS** and Sen, 2025]

2. Logarithmic soft-graviton theorem in $d > 4$

Derive the gravitational logarithm and its classical waveform, including nonlinear gravitational sources.

BS, to appear

3. A celestial interpretation

The $\omega^{d-4} \ln \omega$ term gives a double Mellin pole at $\Delta = 4 - d$, with a pairwise coefficient $\sum_b \sum_{a \neq b} (\cdots)$.

Can additional celestial fields turn this double particle sum into a local Ward identity on S^{d-2}, with a higher-dimensional dipole current?

$d = 4$: [Banerjee, Mandal and **BS**, 2026]

Thank you

The talk ends. The long-range tail continues.

IR finite $\not\Rightarrow$ log-free

Part V

Extra Slides

$d = 4$: Logarithmic trajectory and logarithmic soft theorem

1. Long-range forces accelerate particles even at asymptotically late times:

The b-th outgoing particle (proper time τ_b) trajectory

$$m_b \ddot{X}_b^\mu = q_b \sum_{a \neq b} F_{(a)}^{\mu\nu}(X_b) \dot{X}_{b\nu}, \qquad X_b^\mu = r_b^\mu + v_b^\mu \tau_b + Y_b^\mu(\tau_b).$$

$$F_{(a)}(X_b) \sim \tau_b^{-2} \;\Rightarrow\; \ddot{Y}_b \sim \tau_b^{-2} \;\Rightarrow\; Y_b \sim \ln \tau_b.$$

Asymptotically the particles accelerate and hence radiate electromagnetic wave.

2. A frequency power series misses late-time radiation but logarithms generate waveform tails:

$$\underbrace{\omega^j \;\longleftrightarrow\; i^j \delta^{(j)}(u)}_{j \geq 0\text{: localized at } u=0} \qquad \underbrace{\ln(\omega \pm i0) \;\longleftrightarrow\; \text{one-sided } 1/u}_{\text{late-/early-time tail}}.$$

[Laddha and Sen, 2018; **BS** and Sen, 2018]

$d = 4$: Logarithmic trajectory and logarithmic soft theorem

1. Long-range forces accelerate particles even at asymptotically late times:

The b-th outgoing particle (proper time τ_b) trajectory

$$m_b \ddot{X}_b^\mu = q_b \sum_{a \neq b} F_{(a)}^{\mu\nu}(X_b) \dot{X}_{b\nu}, \qquad X_b^\mu = r_b^\mu + v_b^\mu \tau_b + Y_b^\mu(\tau_b).$$

$$F_{(a)}(X_b) \sim \tau_b^{-2} \;\Rightarrow\; \ddot{Y}_b \sim \tau_b^{-2} \;\Rightarrow\; Y_b \sim \ln \tau_b.$$

Asymptotically the particles accelerate and hence radiate electromagnetic wave.

2. A frequency power series misses late-time radiation but logarithms generate waveform tails:

$$\underbrace{\omega^j \;\longleftrightarrow\; i^j \delta^{(j)}(u)}_{j \geq 0\text{: localized at } u=0} \qquad \underbrace{\ln(\omega \pm i0) \;\longleftrightarrow\; \text{one-sided } 1/u}_{\text{late-/early-time tail}}.$$

3. Low-frequency waveforms probe QED soft factorization:

The waveform tail indicates the existence of universal logarithmic soft photon theorem.

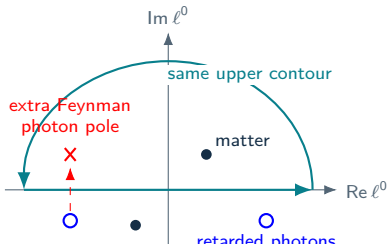
In $d = 4$, the S-matrix is IR divergent. After factoring out common IR factors, ω cuts off the region $\omega \ll |\ell| \ll \Lambda$ that determines the logarithmic soft photon theorem

(ℓ: exchanged loop momentum; Λ: hard scale).

[Laddha and Sen, 2018; **BS** and Sen, 2018]

From the quantum soft factor to classical radiation

1. Compare the integrands: In an equivalent momentum-space calculation of the classical frequency-space waveform, replacing the Feynman photon propagator in $\mathcal{K}^{\rm F}_{ab}$ by the retarded Green's function yields $S_{\rm cl,log} = i\varepsilon_\mu \widehat{J}^\mu_{\rm acc,log}(k)$.



2. Use a contour with no retarded photon pole:

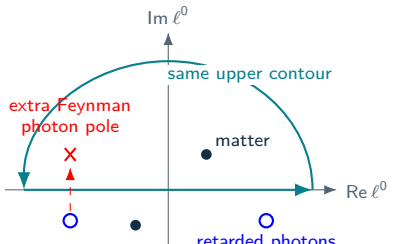
Retarded integral: the enclosed matter residues reproduce the classical current.

Feynman integral: the additional on-shell photon residue gives the quantum remainder.

Four-dimensional comparison: [**BS** and Sen, 2018]

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3. Restore the causal continuation to obtain the waveform:

At $\omega > 0$, both $\ln(\omega \pm i0)$ equal $\ln \omega$. Outgoing and incoming histories fix the $+i0$ and $-i0$ prescriptions, respectively.

The waveform–soft-factor relation and inverse Fourier transform then recover the time-domain tails.

Four-dimensional comparison: [**BS** and Sen, 2018]

Higher dimensions: IR finite, yet logarithmically soft

$$\ddot{Y}_b \sim |\tau_b|^{-(d-2)} \quad \Rightarrow \quad \dot{Y}_b \sim |\tau_b|^{-(d-3)} \quad \Rightarrow \quad Y_b - \text{constant} \sim |\tau_b|^{-(d-4)}.$$

What changes for $d > 4$?

- The correction to straight line trajectory approaches to a constant, unlike $\ln \tau_b$ in $d = 4$.
- Massive QED amplitudes are IR finite: no eikonal IR pole determines the logarithm.

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What survives?

In the Fourier-transformed current $\widehat{J}^\mu$, the $(d-4)$th moment of the velocity tail is marginal:

$$\widehat{J}^\mu_{\text{acc,log}} \sim \omega^{d-4} \int^{\omega^{-1}} \mathrm{d}\tau_b \, \tau_b^{d-4} \, \dot{Y}^\mu_b$$
$$\sim \omega^{d-4} \int^{\omega^{-1}} \frac{\mathrm{d}\tau_b}{\tau_b}.$$

The frequency-space waveform carries an additional $\omega^{(d-4)/2}$ multiplicative factor.

The classical limit of the soft factor, $S_{\text{cl}} = i\varepsilon_\mu \widehat{J}^\mu$, thus predicts a logarithmic soft factor:

$$\boxed{S_{\log} \sim \omega^{d-4} \ln(\omega/\Lambda)}$$

The logarithm moves to a higher soft order; IR finiteness does not remove it.

A finite loop integral can contain a soft logarithm

A radial loop model of the same marginal behaviour, with $n = d - 4 \geq 1$ and $0 < \omega \ll \Lambda$:

$$I_n(\omega) = \int_0^\Lambda \mathrm{d}\ell \, \frac{\ell^n}{\ell + \omega}, \qquad I_n(0) = \frac{\Lambda^n}{n} < \infty.$$

Integrate first: the exact answer is

$$\boxed{I_n(\omega) = \underbrace{\sum_{j=0}^{n-1} \frac{(-\omega)^j \Lambda^{n-j}}{n-j}}_{\text{analytic polynomial}} + (-\omega)^n \ln\left(1 + \frac{\Lambda}{\omega}\right).}$$

Where does the logarithm come from? For $\omega \ll \ell \ll \Lambda$,

$$\frac{\ell^n}{\ell + \omega} = \sum_{j=0}^{n-1} (-\omega)^j \ell^{n-1-j} + \frac{(-\omega)^n}{\ell} + \mathcal{O}\left(\frac{\omega^{n+1}}{\ell^2}\right).$$

$$(-\omega)^n \int_{\sim\omega}^{\Lambda} \frac{\mathrm{d}\ell}{\ell} = (-\omega)^n \ln \frac{\Lambda}{\omega} + \mathcal{O}(\omega^n) \xrightarrow{n=d-4} \boxed{S_{\log} \sim \omega^{d-4} \ln \omega}.$$

IR finiteness does not imply analyticity: the *n*th soft coefficient probes a marginal integral.